Structure of cellular solids

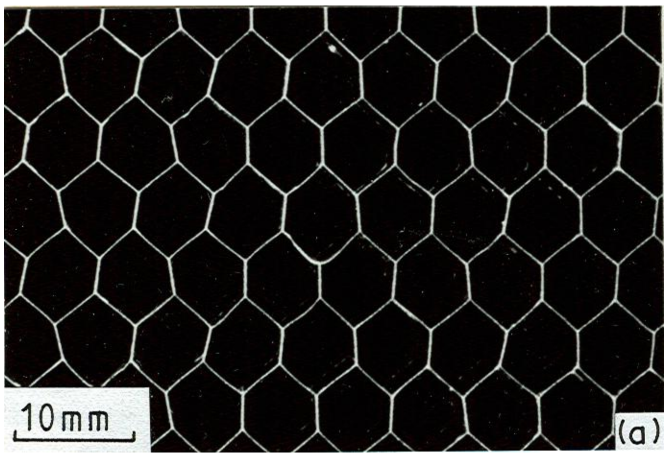
Fig 2.3a — 2D honeycombs: polygonal cells pack to fill 2D plane
prismatic in 3rd direction

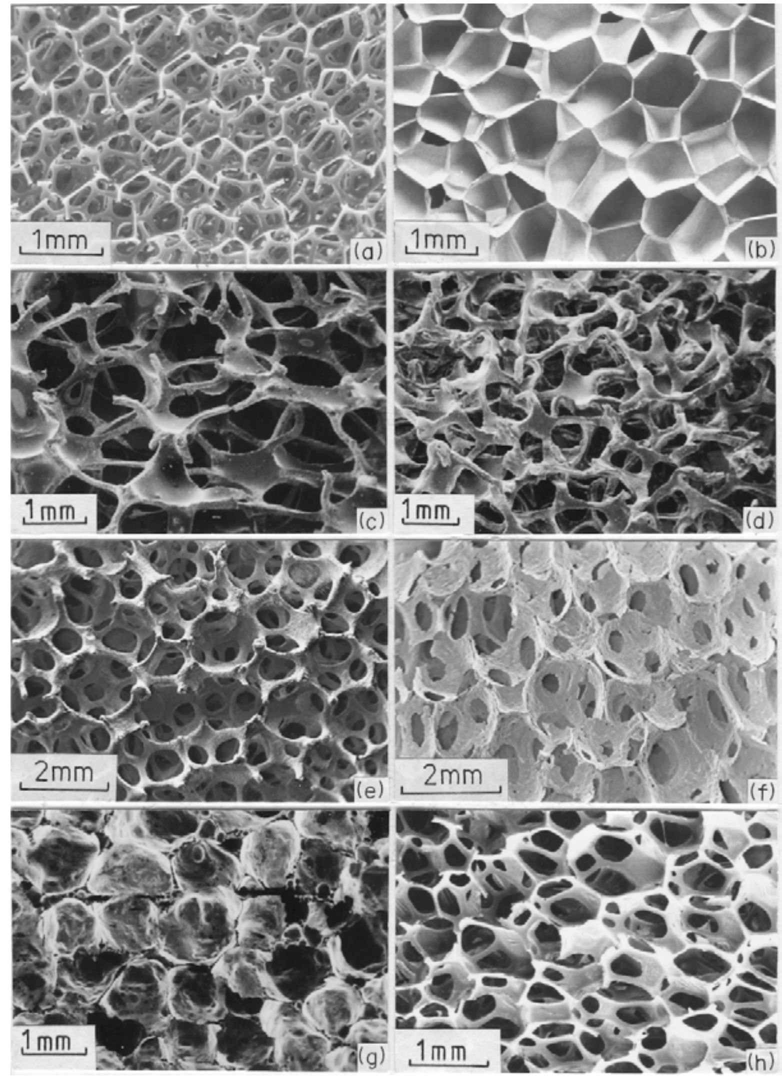
Fig 2.5 — 3D foams: polyhedral cells pack to fill space

Properties of cellular solid depend on:

- properties of solid it is made from (ρ_s, E_s, σ_{ys} ...)
- relative density, ρ^*/ρ_s (= volume fraction solids)
- cell geometry

- cell shape → anisotropy
- foams - open vs. closed cells
  - open: solid in edges only; voids continuous
  - closed: faces also solid; cells closed off from one another
- cell size - typically not impt.





Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Relative density

ρ^* = density of cellular solid $\qquad \rho_s$ = density of solid it is made from

$$\frac{\rho^*}{\rho_s} = \frac{M_s}{V_T}\,\frac{V_s}{M_s} = \frac{V_s}{V_T} = \text{volume fraction of solid}$$
$$(= 1 - \text{porosity}).$$

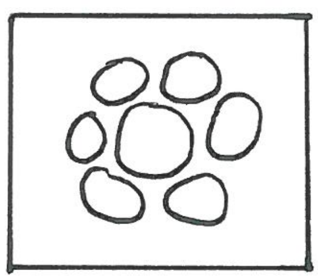
Typical values:

collagen-GAG scaffolds : $\rho^*/\rho_s = 0.005$

typical polymer foams : $0.02 < \rho^*/\rho_s < 0.2$

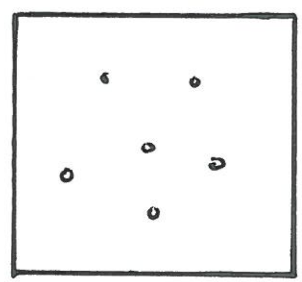
soft woods : $0.15 < \rho^*/\rho_s < 0.4$

- as ρ^*/ρ_s increases, cell edges (+ faces) thicken, pore volume decreases
- in limit → isolated pores in solid



$\rho^*/\rho_s < 0.3$

cellular solid



$\rho^*/\rho_s > 0.8$

isolated pores in solid

Unit cells

Fig 2.11 2D honeycombs: - triangles, squares, hexagons
- can be stacked in more than 1 way
- different number of edges/vertex
- Fig 2.11 (a) (e) isotropic; others anisotropic

3D foams: rhombic dodecahedra + tetrakaidecahedra pack to fill space (apart from △ □ ⬡ prisms)

Fig 2.13 [Greek: hedron = face; do = 2; deca = 10; tetra = 4; kai = and]

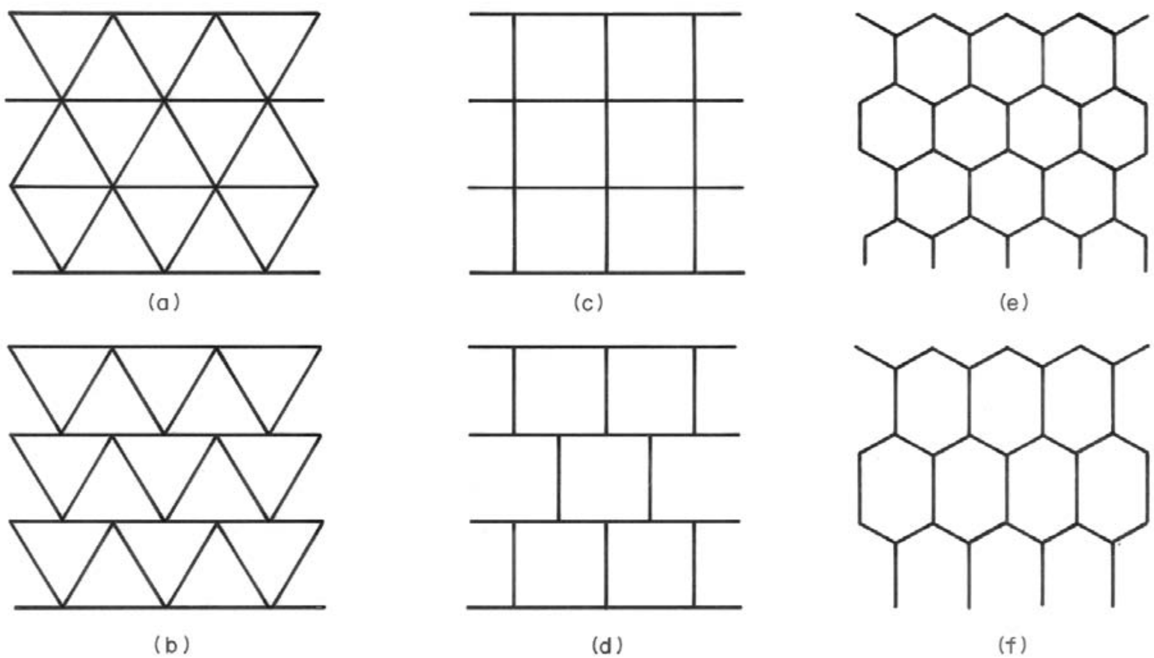
tetrakaidecahedra - bcc packing ; geometries in Table 2.1

- foams often made by blowing gas into a liquid
- if surface tension is only controlling factor & if it is isotropic, then the structure is one that minimizes surface area at constant volume

Fig 2.4 Kelvin (1887): tetrakaidecahedron with slightly curved faces is the single unit cell that packs to fill space + minimizes surface area/volume.

Weaire - Phelan (1994): identified "cell" made up of 8 polyhedra that has slightly lower surface area/volume (obtained using a numerical technique - "surface evolver")

Unit Cells: Honeycombs



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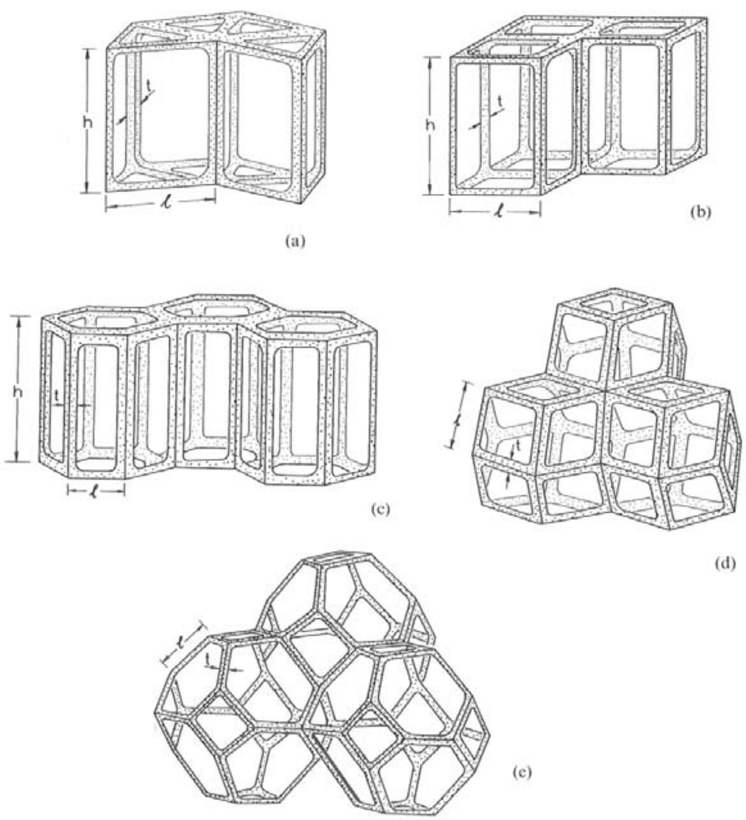
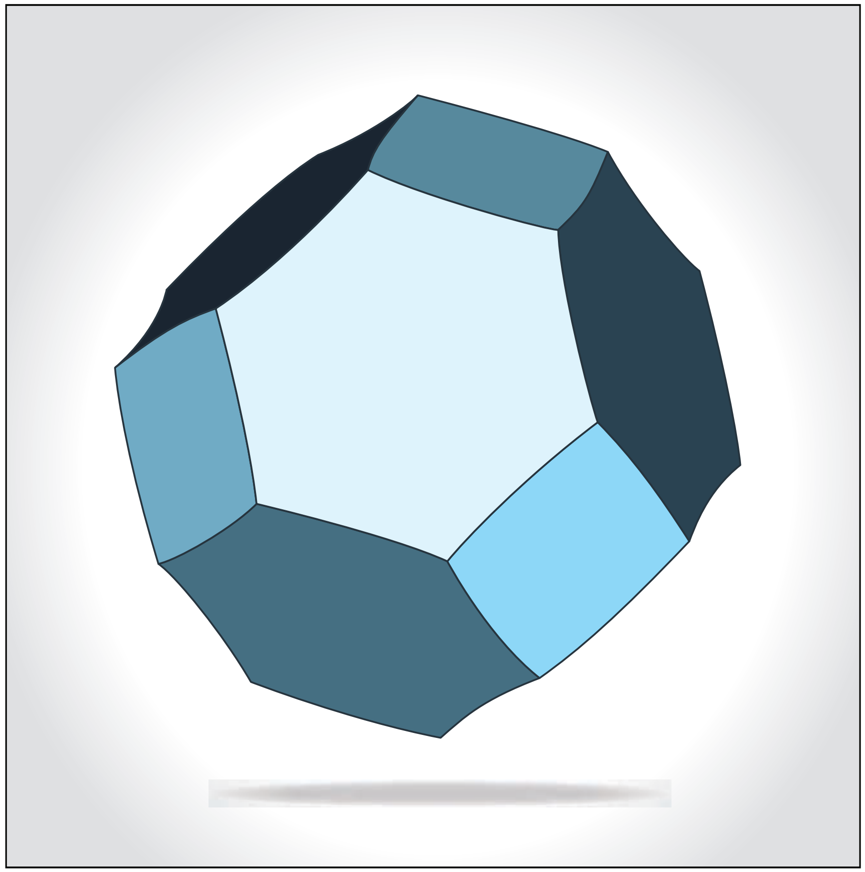
Unit Cells: Foams



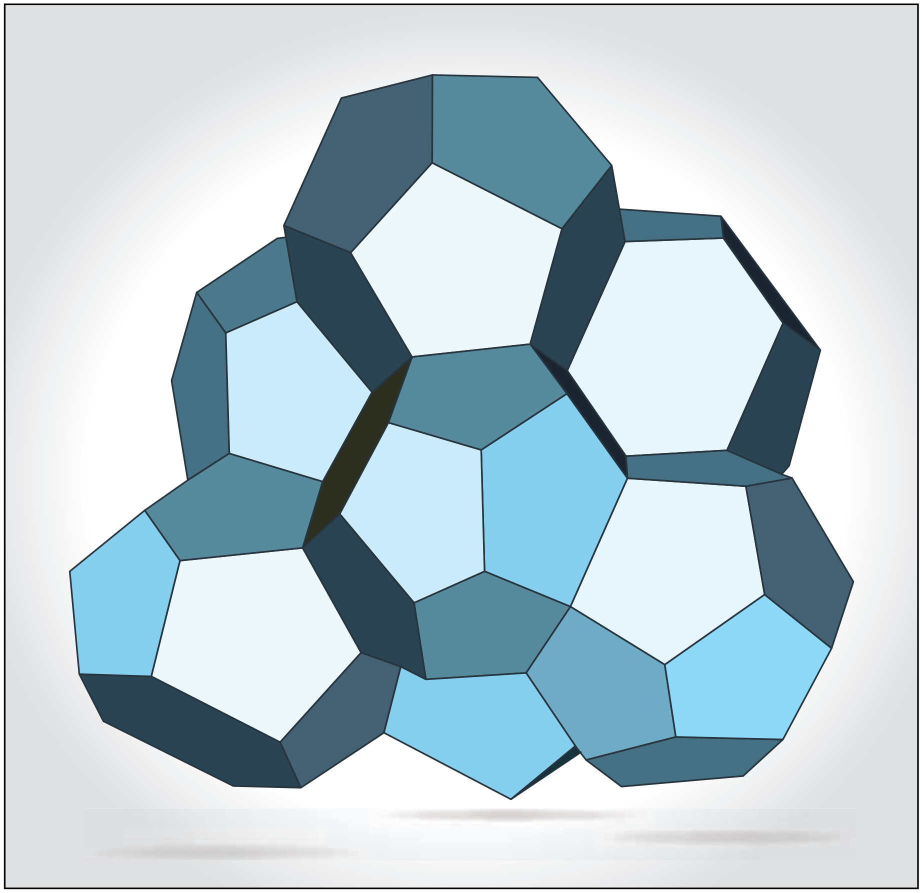
Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Unit Cells:
Kelvin Tetrakaidecahedron



Kelvin's tetrakaidecahedral cell.
Source: Professor Denis Weaire; Figure 2.4 in Gibson, L. J., and M. F. Ashby. *Cellular Solids Structure and Properties*. Cambridge University Press, 1997.

Unit Cells: Weaire-Phelan



Weaire and Phelan's unit cell.
Source: Professor Denis Weaire; Figure 2.4 in Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. Cambridge University Press, 1997.

Voronoi honeycombs + foams

- foams sometimes made by supersaturating liquid with a gas
  + then reducing the pressure, so that bubbles nucleate + grow
- initially form spheres; as they grow, they intersect + form polyhedral cells
- Consider an idealized case: bubbles all nucleate randomly in space at same time + grow at same linear rate
  - obtain Voronoi foam (2D - Voronoi honeycomb)

Fig 2.14a

  - Voronoi structures represent structures that result from nucleation + growth of bubbles

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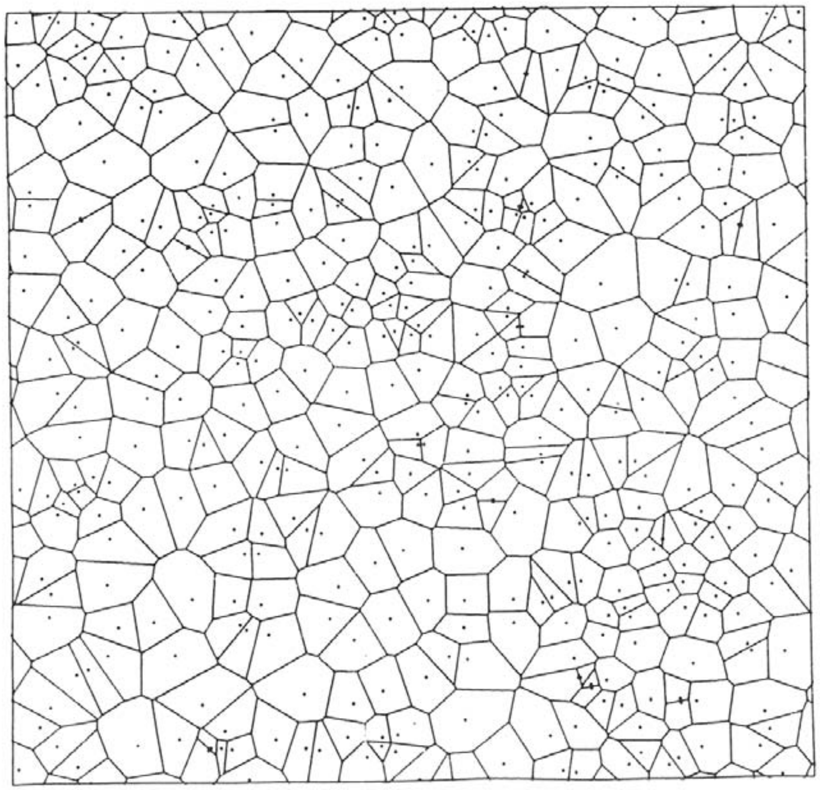
- Voronoi honeycomb is constructed by forming the perpendicular bisectors between random nucleation points & forming the envelope of surfaces that surrounds each point.
- each cell contains all points that are closer to its nucleation point than any other
- cells appear angular
- if specify exclusion distance (nucleation points no closer than exclusion dist.) then cells less angular + of more similar size.

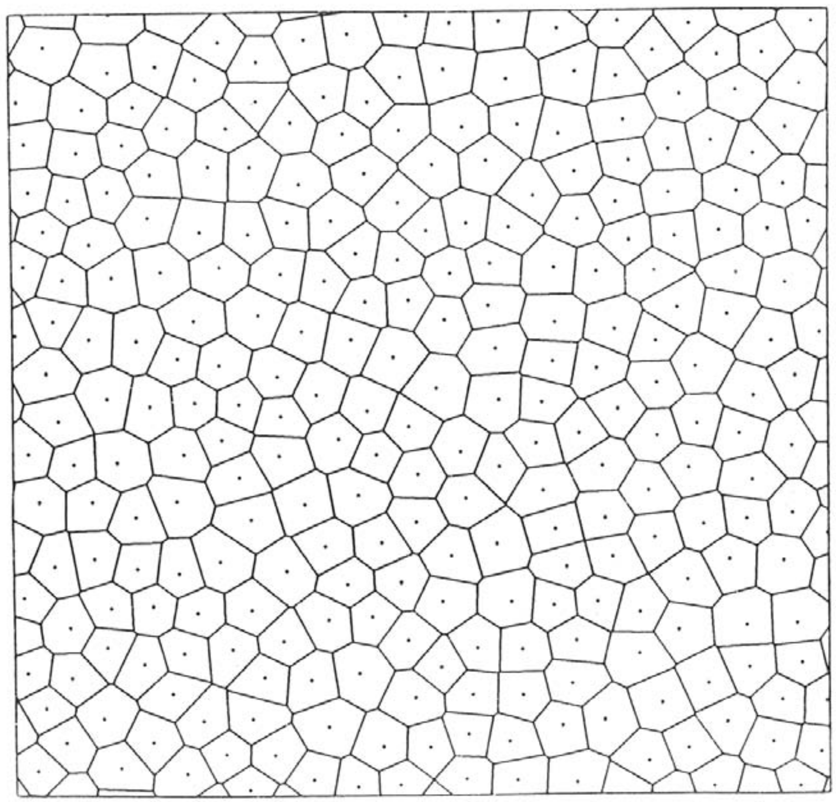
Fig 2.14b

Voronoi Honeycomb



Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

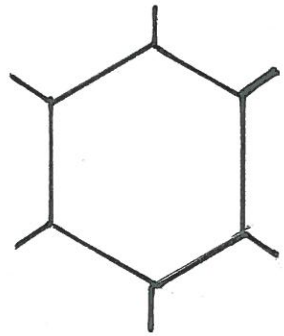
Voronoi Honeycomb with Exclusion Distance



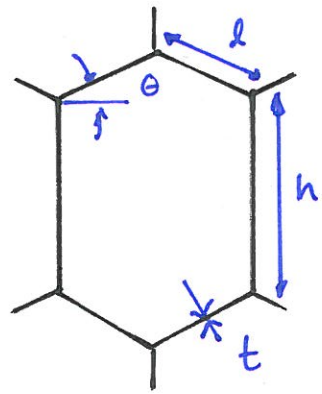
Gibson, L. J., and M. F. Ashby. *Cellular Solids: Structure and Properties*. 2nd ed. Cambridge University Press, 1997. Figure courtesy of Lorna Gibson and Cambridge University Press.

Cell shape, mean intercept length, anisotropy

Honeycombs



regular hexagon:
isotropic in plane



elongated hexagon: anisotropic

h/l, θ define cell shape.

Foams

Huber paper Fig. 9

- characterize cell shape, orientation by mean intercept lengths
- consider circular test area of plane section
- draw equidistant parallel lines at $\theta = 0°$
- count number of intercepts of cell wall with lines

N_c = no. cells per unit length of line

$$L(\theta = 0°) = 1.5/N_c$$

Mean Intercept Length

- increment θ by some amount (eg. 5°) & repeat
- plot polar diagram of mean intercept lengths as $f(\theta)$
- fit ellipse to the points (in 3D, ellipsoid)
- principal axes of ellipsoid ⇒ principal dimensions of cell
- orientation of ellipse corresponds to orientation of cell
- eq'n of ellipsoid: $Ax_1^2 + Bx_2^2 + Cx_3^2 + 2Dx_1x_2 + 2Ex_1x_3 + 2Fx_2x_3 = 1$

- write as matrix $M = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}$

- can also represent as tensor "fabric tensor"
- if all non-diagonal elements of the matrix are zero then diagonal elements correspond to principal cell dimensions.

Connectivity

- vertices connected by edges which surround faces which enclose cells
- edge connectivity, Z_e = no. edges meeting at a vertex

  typically $Z_e = 3$ for honeycombs
  $Z_e = 4$ for foams

- face connectivity, Z_f = no. faces meeting at an edge

  typically, $Z_f = 3$ for foams

Euler's law

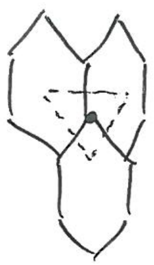
- total number of vertices, V, edges, E, faces, F & cells, C related by Euler's law (for a large aggregate of cells)

2D: $F - E + V = 1$

3D: $-C + F - E + V = 1$

For an irregular, 3-connected honeycomb (with cells with different # edges)

what is average no. sides / face, $\bar{n}$?



$Z_e = 3 \quad \therefore \quad E/V = 3/2$ (each edge shared between 2 vertices)

if F_n = no. faces with n sides, then

$$\sum \frac{n F_n}{2} = E$$ (factor of 2 since each edge separates 2 faces)

Using Euler's law:

$$F - E + \frac{2}{3} E = 1$$

$$F - \frac{1}{3} \sum \frac{n F_n}{2} = 1$$

$$6F - \sum n F_n = 6$$

$$6 - \frac{\sum n F_n}{F} = \frac{6}{F}$$

as F becomes large, RHS $\to 0$

$$\frac{\sum n F_n}{F} = \text{avg. no. sides per face, } \bar{n}$$

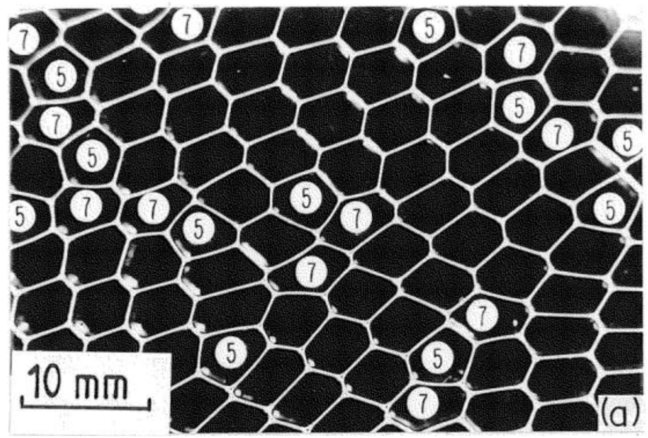
$$\bar{n} = 6$$

For 3-connected honeycomb,

avg. # sides <u>always</u> 6

Fig 2.9a

Euler's Law



Soap Honeycomb

Aboav - Weaire law

- Euler's law: for 3-connected honeycomb, avg. no sides/face = 6
- introduction of a 5-sided cell requires introduction of 7-sided cell etc
- generally, cells with more sides (in 2D) (or faces, in 3D) than average,

  have neighbors with fewer sides (in 2D) (or faces, in 3D) than average
- Aboav - observations in 2D soap froth
  Weaire - derivation

- 2D: If a candidate cell has n sides, then the average number of sides of its n neighbors is $\bar{m}$

$$\bar{m} = 5 + \frac{6}{n} \qquad (2D)$$

Lewis' rule

- Lewis examined biological cells & 2D cell patterns
- found that area of a cell varied linearly with the number of its sides

$$\frac{A(n)}{A(\bar{n})} = \frac{n - n_0}{\bar{n} - n_0}$$

$A(n)$ = area of cell with n sides

$A(\bar{n})$ = " " " " avg. no. sides, $\bar{n}$

n_0 = constant (Lewis found $n_0 = 2$)

- holds for Voronoi honeycombs; Lewis found holds for most othe 2D cells

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- also, in 3D:

$$\frac{V(f)}{V(\bar{f})} = \frac{f - f_0}{\bar{f} - f_0}$$

$V(f)$ = volume of cell with f faces

$V(\bar{f})$ = " " " " avg. no. faces $\bar{f}$

f_0 = constant ~ 3

Modelling cellular solids - structural analysis

3 main approaches:

(1) unit cell eg. honeycomb - hexagonal cells
foam - tetrakaidecahedra (but cells not all tetrakaidecahedra)

(2) dimensional analysis

foams - complex geometry, difficult to model exactly
- instead, model mechanisms of deformation + failure
(do not attempt to model exact cell geometry)

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(3) finite element analysis

- can apply to random structures (eg. 3D Voronoi) or to
micro-computed tomography information (eg. trabecular bone)
- most useful to look at local effects
(e.g. defects - missing struts - osteoporosis
size effects)