APPENDIX (LECTURE #2) :

I. Review / Summary of Cantilever Beam Theory
II. Summary of Harmonic Motion
III. Limits of Force Detection
IV. Excerpts from *Vibrations and Waves*, A.P. French, W. W. Norton and Company, 1971

I. Review / Summary of Cantilever Beam Theory from 3.032 [1]

A *cantilevered* beam is one that is fixed at one end and free at the opposite end, as shown in Figure 1.

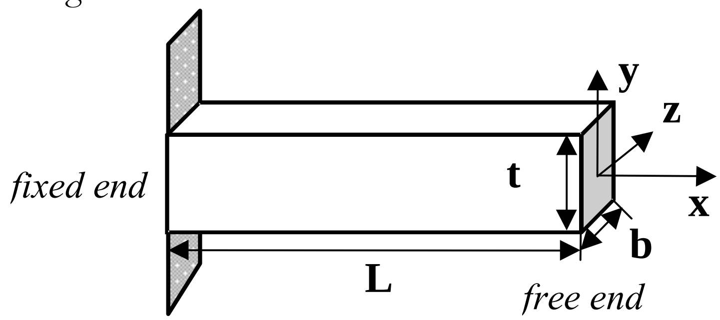


Figure 1. Nomenclature for a cantilevered beam with rectangular cross section ; L=length or span (m), b=width (m), t=height or thickness (m), I=moment of inertia of cross-sectional area (m^4), E=Young's (elastic) modulus (Pa=N/m^2), and EI=flexural modulus (Nm^2)

Consider the case where a concentrated force is applied in the downwards direction at the free end of a cantilever (Figure 2.).

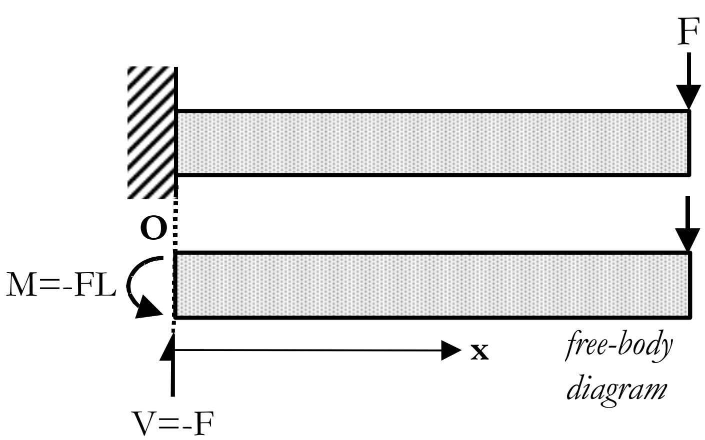


Figure 2. A loaded, cantilevered beam and corresponding free-body diagram

A free-body diagram of the beam shows that a reactant shear force, V, and a reactant bending moment M, must exist in order to maintain static equilibrium. By taking the conditions for equilibrium one finds that :

$$\sum F_y = 0 = V + F \Rightarrow V = -F \quad \textbf{(1)}$$

$$\sum M_o = 0 = M + FL \Rightarrow M = -FL \quad \textbf{(2)}$$

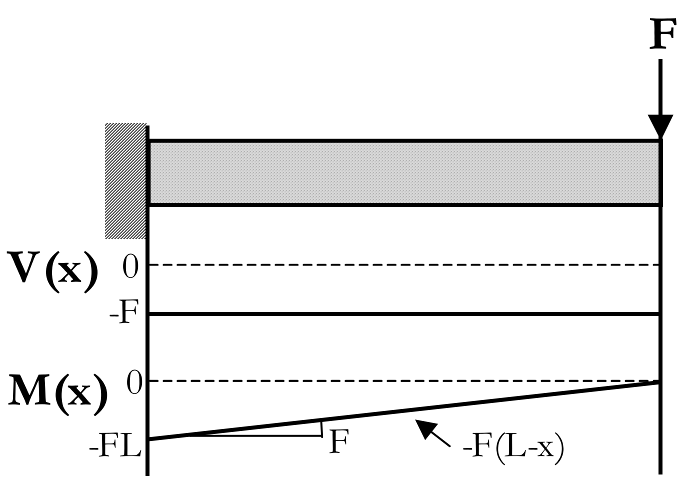
No matter where a transverse cut is taken along the beam and a free-body diagram constructed, the magnitude of the shear force, V, is found to be constant and equal to F throughout the length of the beam:

$$V(x) = F = constant \quad \textbf{(3)}$$

Since $V(x) = -\frac{dM}{dx}$, the moment, *M(x)*, varies linearly from a maximum of zero at the free end to a minimum of -FL at the wall. Hence, *M(x)* is linear and equal to :

$$M(x) = -F(L-x) \quad \textbf{(4)}$$

Equations *(3)* and *(4)* are shown graphically in Figure 3.



<u>Figure 3.</u> Shear and bending moment diagrams for the cantilevered beam given in Figure 2.

The equation for the slope of the y-displacement curve, *θ(x)*, is defined as follows:

$$\theta(x) = \frac{1}{EI}\int_0^x M(x)dx \quad \textbf{(5)}$$

Substituting equation *(4)* into equation *(5)* we obtain :

$$\theta(x) = -\frac{1}{EI}\int_0^x F(L-x)dx = -\frac{1}{EI}\int_0^x (FL - Fx)dx$$

$$\theta(x) = -\frac{1}{EI}\left[(FLx) - \frac{Fx^2}{2}\right] + C_1 \quad \textbf{(6)}$$

The integration constant, C_1, can be obtained from the boundary condition that the slope of y-displacement curve, $\theta(x)$, must be zero at the wall (x=0):

$$\theta(0) = 0 = -\frac{1}{EI}\left[(FL0) - \left(\frac{F0^2}{2}\right)\right] + C_1 \Rightarrow C_1 = 0$$

$$\theta(x) = -\frac{1}{EI}\left[(FLx) - \left(\frac{Fx^2}{2}\right)\right] \quad \textbf{(7)}$$

The equation for the y-displacement curve or elastic curve, $y(x)$, can be found as follows:

$$y(x) = \int_0^x \theta(x)dx \quad \textbf{(8)}$$

Substituting equation *(7)* into equation *(8)* we obtain :

$$y(x) = -\int_0^x \frac{1}{EI}\left[(FLx) - \left(\frac{Fx^2}{2}\right)\right]dx$$

$$y(x) = -\frac{1}{EI}\left[\left(\frac{FLx^2}{2}\right) - \left(\frac{Fx^3}{6}\right)\right] + C_2 \quad \textbf{(9)}$$

The integration constant, C_2, can be obtained from the boundary condition that the y-displacement $y(x)$ must be zero at the wall (x=0) :

$$y(x) = 0 = -\frac{1}{EI}\left[\left(\frac{FL0^2}{2}\right) - \left(\frac{F0^3}{6}\right)\right] + C_2 \Rightarrow C_2 = 0$$

$$y(x) = -\frac{F}{EI}\left[\left(\frac{Lx^2}{2}\right) - \left(\frac{x^3}{6}\right)\right] \quad \textit{(10)}$$

The maximum deflection occurs at the free end of the cantilever and can be found by substituting $x=L$ into equation *(10)* :

$$y_{max}(x = L) = -\frac{FL^3}{3EI} \quad \textit{(11)}$$

Equations *(10)* and *(11)* are shown graphically in Figure 4.

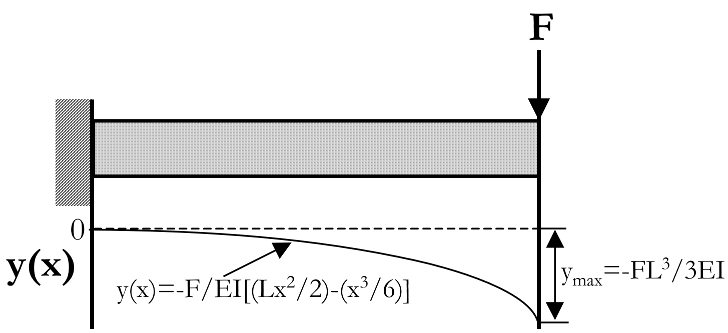


Figure 4. Elastic curve of cantilevered beam

By rearranging equation *(11)*, one can obtain the applied load as a function of the deflection at the end of the beam:

$$F = \left(-\frac{3EI}{L^3}\right) y_{max} \quad \textit{(12)}$$

Here, we see that the applied force is directly proportional to the displacement at the end of the beam and hence, the cantilever can be represented by a linear elastic, Hookean spring (Figure 5.):

$$F = k\delta \quad \textit{(13)}$$

where $\delta = y_{max}$ is the maximum deflection at the end of the cantilever (force spectroscopy notation), and k is the "cantilever spring constant" :

$$k = -\frac{3EI}{L^3} \quad \textbf{\textit{(14)}}$$

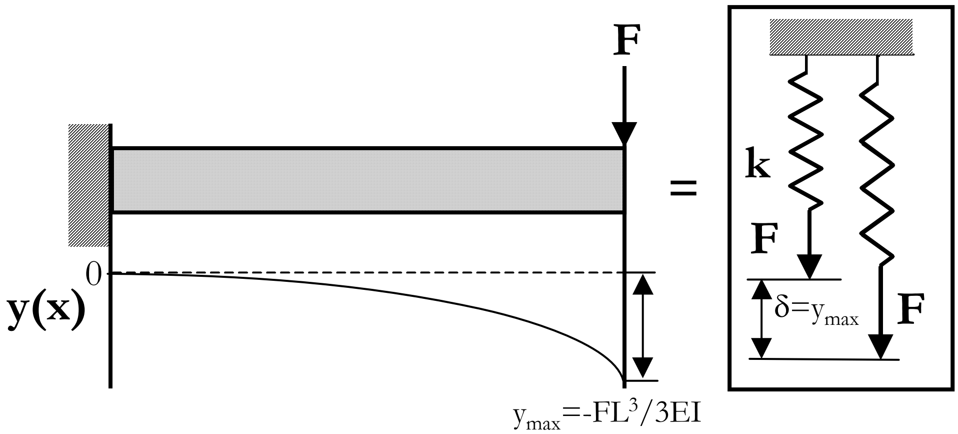


Figure 5. Representation of cantilevered beam by a linear elastic, Hookean spring

Hence, k is a function only of the beam dimensions and the elastic modulus.

Typically, V-shaped cantilevers are used for high-resolution force spectroscopy experiments (Figure 6.).

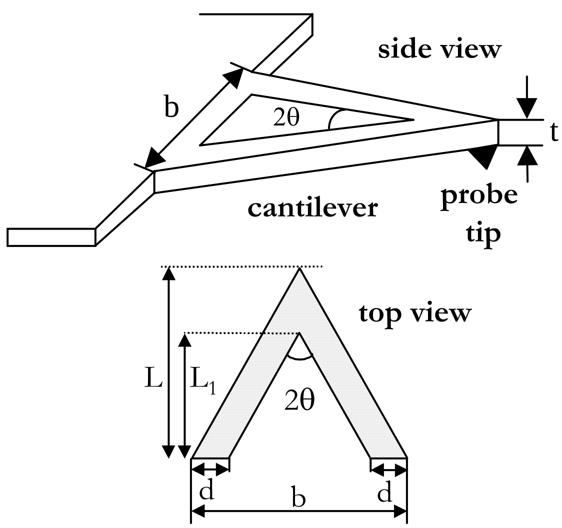


Figure 6. Dimensions of a V-shaped cantilever beam

Table I. displays approximate formulas for the k of V-shaped cantilevers.

Table I. Formulas for the k of V-shaped cantilevers [2].

Reference	Cantilever Spring Constant, k	% error
[2]	$\frac{Et^3d}{2L^3}\left[1+\frac{b^2}{4L^2}\right]^{-2}$	25
[3]	$\frac{0.5Et^3d}{L^3}$	16
[4]	$\frac{Et^3d}{2L^3}\left[1+\frac{4d^3}{b^3}\right]^{-1}$	13
[4]	$\frac{Et^3d}{2L^3}\cos\theta\left[1+\left(\frac{4d^3}{b^3}\right)(3\cos\theta-2)\right]^{-1}$	2

References :

[1] *Mechanics of Materials*, D. Roylance, John Wiley and Sons, Inc. 1996.
[2] T. R. Albrecht, S. Akamine, T. E. Carver, and C. F. Quate, *J. Vac. Sci. Tech.* **A8**, 3386 (1990).
[3] H.-J. Butt, P. Siedle, K. Siefert, K. Fendler, T. Seeger, E. Bamberg, A. L. Weisenhorn, K. Goldie, and A. Engel, *J. Microscopy* **169**, 75 (1993).
[4] J. E. Sader, *Rev. Sci. Instrum.* **66** (9), 4583 (1995).

II. Summary of Harmonic Oscillators

(*reference : *Vibrations and Waves*, A. P. French, W. W. Norton and Company, NY 1971.)

II.A. Free Vibrations

Basic Physics Equations :

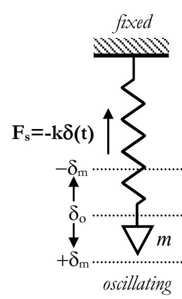
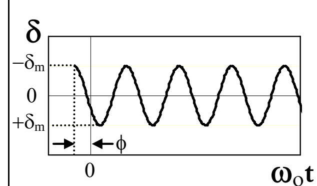
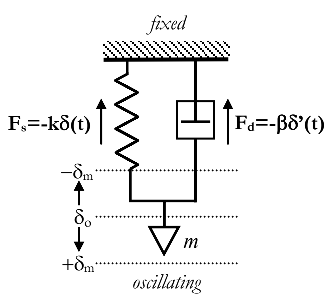
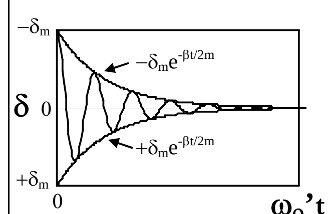
$\delta(t)$=displacement(m)

$v(t)$=velocity(m/s)=$d\delta(t)/dt=\delta'(t)$

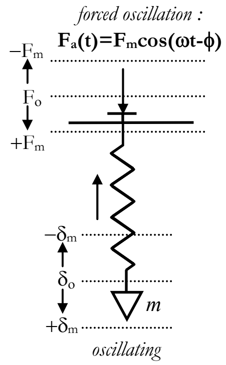
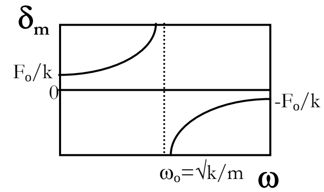
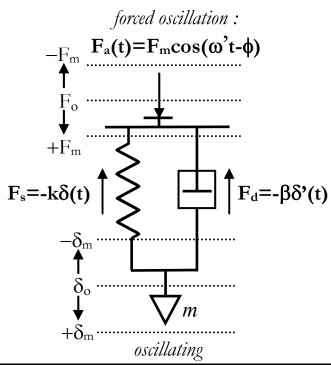
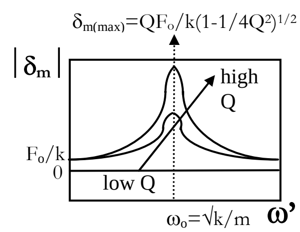
$a(t)$=acceleration(m/s^2)= $d^2\delta(t)/dt^2=\delta''(t)$

$F(t)$=force(N)=$ma(t)$ where : m=mass(g)

$U(\delta)$=potential energy(Nm)=$\int F(\delta)d\delta$

Type of Harmonic Motion :	Model Schematic :	Equations of Motion :	Solutions to Equations of Motion :
Simple Harmonic Motion (SHM) : ν=natural or resonant frequency (Hz=1 oscillation/s=s^{-1}) ϖ=natural or resonant angular frequency=$2\pi\nu$ (rad/s^{-1}) δ_m=displacement amplitude (m) ϕ=phase constant $\varpi t+\phi$=phase F_s=spring recovery force k=spring constant (N/m)		$ma=F_s \Rightarrow$ $m\delta''(t)+k\delta(t)=0$	$\delta(t)=\delta_m\cos(\varpi_o t-\phi)$ $\varpi_o^2=k/m$ 
Damped Harmonic Motion (DHM) : β=damping (viscosity) coefficient F_d=dashpot or dissipative force ϖ_o'=natural or resonant angular frequency for a damped system (rad/s^{-1}) Q=quality factor		$ma=F_s+F_d \Rightarrow$ $m\delta''(t)+\beta\delta'(t)+k\delta(t)=0$	$\delta(t)=\delta_m e^{-\beta t/2m}\cos(\varpi_o' t-\phi)$ $\varpi_o'=\sqrt{[(k/m)-(\beta^2/4m^2)]}$  $Q^2=km/\beta^2$

II.B. Forced Vibrations

Type of Harmonic Motion :	Model Schematic :	Equations of Motion :	Solutions to Equations of Motion :
Driven Harmonic Motion (DHM) : ϖ= frequency of applied force oscillation (rad/s^{-1}) $\varpi = \omega_o$ "resonance" occurs; maximum amplitude of oscillations, δ_m		$ma=F_s-F_a \Rightarrow$ **$m\delta''(t)+k\delta(t)=F_a(t)$**	$\delta(t)=\delta_m\cos(\varpi t-\phi)$ $\delta_m(\omega)=F_m/(k-m\omega^2)$ 
Driven / Damped Harmonic Motion (DDHM) : ϖ= frequency of applied force oscillation for damped system (rad/s^{-1})		$ma=F_s+F_d-F_a \Rightarrow$ **$m\delta''(t)+\beta\delta'(t)+k\delta(t)=F_a(t)$**	$\delta(t)=\delta_m\cos(\varpi' t-\phi)$ $\delta_m(\omega')=F_m/(k-m\omega'^2)$ 

III. Limits of Force Detection [1-4]

The lower bound of force detection of any force spectroscopy measurement is determined either by the ***resolution*** or ***thermal fluctuations*** of the transducer.[1,2]

Transducer Resolution. Previously, we have shown that a high-resolution force transducer can be represented by a linear elastic, Hookean spring (equation *(13)*). Let's assume that the minimum detectable displacement is a one-atom deflection (δ_{min}=0.1 nm). Substituting this value into equation *(13)* we obtain the minimum detectable force, $\mathbf{F}_{min}$:

$$F_{min} = (0.1 \text{ nm})k \quad \textbf{\textit{(15)}}$$

Thermal Oscillations. In the absence of any externally applied forces, a force transducer in equilibrium with its surroundings will fluctuate due to the nonzero thermal energy at room temperature, $k_BT = 4.1 \bullet 10^{-21}$ Nm, where k_B is the Boltzmann constant = $1.38 \bullet 10^{-23}$ J/K and T is the absolute temperature (room temperature ≈ 295K). If we model the force transducer as a one-dimensional, free harmonic oscillator as shown in Figure 7.

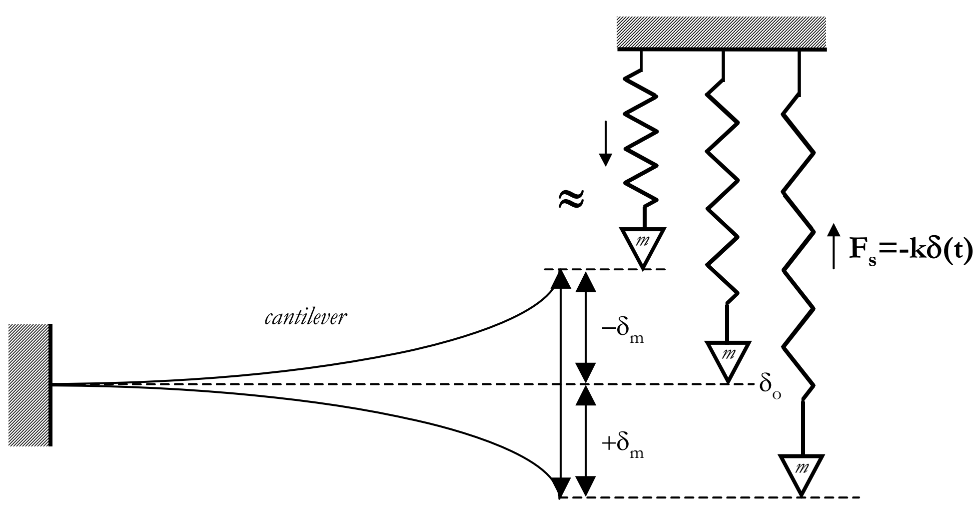


Figure 7. Thermal oscillation of a free cantilever beam

By neglecting higher modes of oscillation and making use of the equipartition theorum, the average root-mean-square (RMS) amplitude of the displacement oscillation, $\langle\delta_m^2\rangle^{1/2}$, can be derived as follows.

The potential energy of a force transducer is

$$U = \int_0^{\delta} F(\delta)d\delta \quad \textbf{\textit{(16)}}$$

Substituting Hooke's law for a free, one-dimensional harmonic oscillator (equation *(13)*) into equation *(17)* and integrating gives

$$U = \int_0^{\delta} k\delta d\delta \Rightarrow U = ½k\delta^2 \quad \textbf{\textit{(17)}}$$

The equipartition theorum states that if a system is in thermal equilibrium, every independent quadratic term in the total energy has a mean value equal to $½k_BT$. Hence,

$$U = ½k\,\delta_m{}^2 = ½k_BT \quad \textbf{\textit{(18)}}$$

where δ_m is the amplitude of the displacement oscillation (Figure 7.). Rearranging equation *(1)* and solving for δ_m we obtain

$$\left\langle \delta_m^2 \right\rangle^{1/2} = \sqrt{\frac{k_BT}{k}} \quad \textbf{(19)}$$

where : <> denotes a statistical mechanical average over time. Substituting eq. *(19)* into Hooke's Law, equation *(13),* gives the equation for the RMS amplitude fluctuations in force:

$$\left\langle F_m^2 \right\rangle^{1/2} = \sqrt{\frac{k_BT}{k}} \quad \textbf{\textit{(20)}}$$

A more precise formulation can be derived for a damped harmonic oscillator[5]:

$$\left\langle F_m^2 \right\rangle^{1/2} = \sqrt{\frac{4k_BTkB}{w_o'Q}} \quad \textbf{\textit{(21)}}$$

where B is the measured bandwidth (s^{-1}), Q is the quality factor $=(km)^{1/2}/\beta$, m is the mass (Ns^2/m), β is the damping coefficent (Ns/m), w_o' is the resonant frequency for a damped system (s^{-1}), and k is the transducer spring constant (N/m).

References :

[1] E. Evans, K. Ritchie, and R. Merkel, *Biophys. J.* **1995**, *68*, 2580.
[2] *Nanosystems : Molecular Machinergy, Manufacturing, and Computation*, K. Eric Drexler, John Wiley and Sons, 1992.
[3] J. L. Hutter, Bechhoefer, J. *Rev. Sci. Instrum.* **1993**, *64*, 1868.
[4] H.-J. Butt, P. Siedle, K. Seifert, K. Fendler, T. Seeger, E. Bamburg, A. L. Weisenhorn, K. Goldie, and A. Engel *J. Microsc.* **1993**, *169*, 75-84.
[5] D. Sarid, *Scanning Force Microscopy*, Oxford University Press, p. 48