

Pset 4 solutions, 3.044, 2013

Problem I

Solve the 1D Navier-Stokes equation:

$$\frac{dv_x}{dt} = \nu \frac{d^2 v_x}{dy^2} + \frac{F_x}{\rho}$$

$$\text{steady state} \rightarrow \frac{dv_x}{dt} = 0$$

$$F_x = g \rho \sin \theta$$

θ depends on the local geometry - leave it general for now.

$$\frac{d^2 v_x}{dy^2} = \frac{-g}{\nu} \sin \theta$$

integrate:

$$\frac{dv_x}{dy} = -(g/\nu) \sin \theta y + A$$

B.C.: at $y=L$, $\frac{dv_x}{dy} = 0$. This is a realistic boundary condition for an "infinite" liquid, but we'll approximate the container as finite. Therefore, $A = (g/\nu) \sin \theta L$

$$\frac{dv_x}{dy} = (g/\nu) \sin \theta (L - y)$$

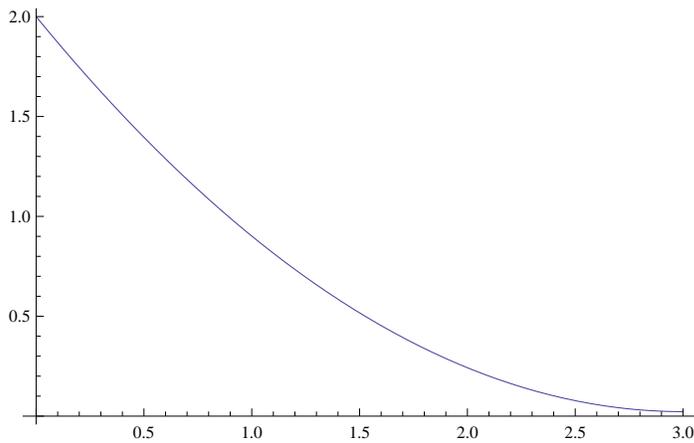
integrate:

$$v_x = (g/\nu) \sin \theta (Ly - y^2/2) + B$$

B.C.: at $y=0$, $v_x = v_0$. Therefore, $B = v_0$.

$$v_x = (g/\nu) \sin \theta (Ly - y^2/2) + v_0$$

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Plot[-Sin[.455] (3 y - y^2 / 2) + 2, {y, 0, 3}]
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Note: there are several ways to do this problem. So long as your boundary condition choices were reasonable, that is okay. Also, the jury is still out whether the buoyant force should be included or not here, because the fluid is self-supporting. Either way, if you explained yourself and your logic is internally consistent, that is okay.

Problem 2

Begin with the ratio of thermal resistances to see where the gradients are:

$$\frac{L_{\text{steel}}/k_{\text{steel}}}{L_{\text{zinc}}/k_{\text{zinc}}} = \frac{(2 \times 10^{-3} / 2 \text{ m}) / (37 \text{ W/mK})}{(50 \times 10^{-6} \text{ m}) / (112 \text{ W/mK})} = 60 > 10$$

⇒ the coating is too thin and conductive to sustain a thermal gradient.

h for a solid-solid interface is $\sim 4000 \text{ W} / \text{m}^2 \text{ K}$, while it is $\sim 10 \text{ W} / \text{m}^2 \text{ K}$ for the air side. We were told that the steel is held at room temperature, so the Zn could lose heat to either the air or the steel. Given that $h_{\text{steel interface}} \gg h_{\text{air interface}}$, we can neglect heat loss to the air. This means that our geometry looks just like a casting, with a steel mold and $L/2 = 50 \mu\text{m}$!

Next, we want to know if the steel can carry heat away quickly (so it looks like an actively-cooled mold), or if heat accumulates in the steel (so it looks like a thick mold), i.e., is it interface-resistance limited, or conduction in the mold-limited?

Biot number for steel:

$$h_{\text{interface}} L_{\text{steel}} / k_{\text{steel}} = (4000 \text{ W} / \text{m}^2 \text{ K}) (0.001 \text{ m}) / (37 \text{ W/mK}) = 0.108$$

This is just over our 0.1 threshold, but because the steel is nearly gradient-free, we should pick interface-limited instead of conduction-limited.

So, we have a thin liquid film with heat loss across a resiting interface into a solid with no gradients = a thin casting with a cooled mold and interface resistance is limiting = die casting.

To solve:

heat of fusion = interface resistance

$$\rho H_f \frac{ds}{dt} = h_{\text{interface}} (T_m - T_{\text{air}})$$

solve, with the B.C. that $s=0$ at $t=0$:

$$s = \frac{h}{H_f \rho} (T_m - T_{\text{air}}) t$$

when $s=50 \mu\text{m}$, it will be solidified.

$T_m=693 \text{ K}$, $T_{\text{air}} = 300 \text{ K}$, $H_f= 112 \text{ kJ/kg}$, $\rho = 7140 \text{ kg} / \text{m}^3$, so the time to solidify is:
 $t = 0.025 \text{ seconds}$

Problem 3

1. Coordinate system: spherical is appropriate here, with the origin at the center of the sphere. It would make sense to define ϕ relative to the flow direction (vertical). Cylindrical works too, you just need different boundary conditions.

2. The problem to be solved is the full form of the Navier-Stokes Equation:

$$\frac{\partial \vec{v}}{\partial t} = \nu \nabla^2 \vec{v} + \frac{\vec{F}}{\rho} - \frac{\nabla P}{\rho}$$

Initial condition:

We will assume steady state, so $\frac{\partial \vec{v}}{\partial t}=0$.

Boundary conditions: we need 6 B.C.'s, 2 for each axis:

1. at $r=R$, $\vec{v} = 0$.
2. at $r \rightarrow \infty$, $\vec{v} = v_{\text{flow}}$.
3. the speed of the flow, v , at $(r, \theta, \phi) = v$ at $(r, \theta, \pi-\phi)$ (a continuity B.C.)
4. independent of θ , i.e., rotational symmetry: v at $(r, \theta_1, \phi) = v$ at (r, θ_2, ϕ) where 1 and 2 are any θ values

5. $\vec{v}(r, \theta, \phi) = \vec{v}(r, \pi+\theta, \pi-\phi)$ (this is an inversion symmetry)
6. at $\phi=\pi/2$, the $v_r = 0$ and $v_\theta = 0$ (geometry: the flow points down around the middle of the particle)

or

Alternatively:

1. at $r=R$, $\vec{v} = 0$.
2. at $r \rightarrow \infty$, $\vec{v} = v_{\text{flow}}$.
3. as $r \rightarrow \infty$, $\frac{\partial \vec{v}}{\partial r} = 0$.
4. as $r \rightarrow \infty$, $\frac{\partial \vec{v}}{\partial \theta} = 0$.
5. as $r \rightarrow \infty$, $\frac{\partial \vec{v}}{\partial \phi} = 0$.
6. Any of the above B.C.'s not already included (B.C.'s #3-6 okay).

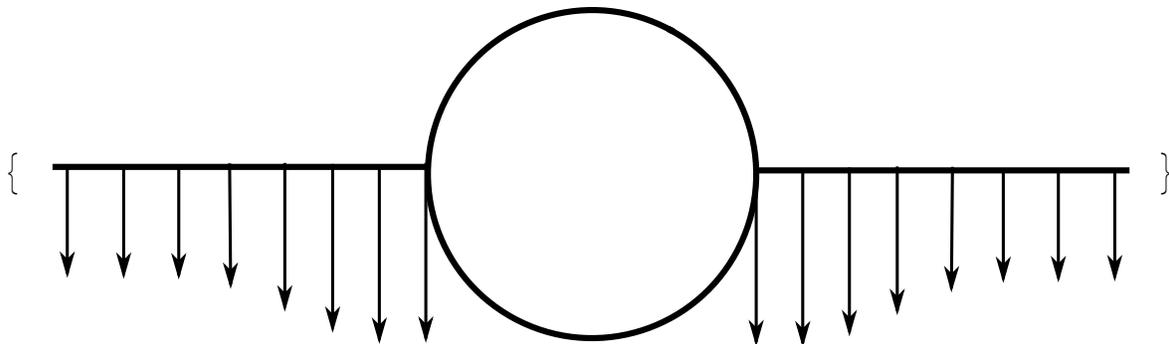
Problem 4

What is different?

The difference between vapor and solid is that there is a “no shear” condition for a fluid-vapor interface, whereas there is a “no slip” condition for a fluid-solid interface. So, B.C. #1 listed above no longer applies, and is instead replaced by

$$\frac{\partial |\vec{v}|}{\partial r} = 0 \text{ at } r=R.$$

The drawing will look more like this



because the shear has to go to zero near the bubble, so the velocity gradient is flat near the bubble surface, and to conserve flow volume, it should flow faster at

the bubble surface than the mean flow because the path it has to take is longer (Bernoulli's principle).

Problem 5

This problem may seem a bit nasty at first because the velocity points in the angular direction, but if you step back and think about it, here's the big picture: you have a momentum source at the inner radius, and a momentum sink at the outer radius. That's it! It is a radial geometry, no nasty θ terms required.

Navier stokes:

$$\frac{\partial v_\theta}{\partial t} = \mu/\rho \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right)$$

There are no body forces, so only the viscous term.

The shear mixer is operating at steady-state:

$$0 = \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right)$$

integrate:

$$\int 0 \, dr = \int \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) dr$$

$$A = r \frac{\partial v_\theta}{\partial r}$$

separate variables:

$$A/r \, \partial r = \partial v_\theta$$

integrate again:

$$A \ln(r) + B = v_\theta$$

Boundary conditions:

$$1. \text{ At } r_{\text{in}}, v_\theta = v_0.$$

$$2. \text{ At } r_{\text{out}}, v_\theta = 0.$$

Therefore,

$$A \ln(r_{\text{in}}) + B = v_0$$

and

$$A \ln(r_{\text{out}}) + B = 0$$

Do some algebra to arrive at:

$$\frac{v_\theta}{v_0} = \frac{\ln(r/r_{\text{out}})}{\ln(r_{\text{in}}/r_{\text{out}})}$$

This problem is identical to the problem we solved in lecture 2, 1D heat transfer through a pipe wall with a hot fluid inside and a cold ambient outside.

The solution was:

$$\Theta = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}$$

With boundary conditions:

$$\text{at } r = r_1, T = T_1.$$

at $r = r_2$. $T = T_2$.

Translation to fluid flow:

$$T \rightarrow v_\theta$$

$$T_1 \rightarrow 0$$

$$T_2 \rightarrow v_0$$

$$r_1 \rightarrow r_{\text{out}}$$

$$r_2 \rightarrow r_{\text{in}}$$

$$\Theta = \frac{T - T_1}{T_2 - T_1} \rightarrow \frac{v_\theta - 0}{v_0 - 0} = \frac{v_\theta}{v_0}$$

Problem 6

(a) This one is essentially out of the reading, but it is a useful exercise to do at some point, because particle settling is a very important concept in materials science. It is how you analyze the stability of a suspension.

All you have to do is balance the gravity force pulling down on a particle against the drag force resisting the sinking of the particle, and the buoyant force of the fluid supporting the particle.

$$F_{\text{gravity}} = m g = V \rho_{\text{particle}} g$$

$$F_{\text{buoyant}} = V \rho_{\text{fluid}} g$$

$$F_{\text{drag}} = f_{\text{laminar}} A K$$

$$\text{let } \Delta\rho = (\rho_{\text{particle}} - \rho_{\text{fluid}})$$

$$F_{\text{gravity}} = F_{\text{kinetic}} + F_{\text{buoyant}}$$

$$V \Delta\rho g = f_{\text{laminar}} A K$$

$$V = (4/3) \pi R^3$$

$$\& \text{ from notes, } f_{\text{laminar}} A K = 6 \pi \mu v R$$

$$(4/3) \pi R^3 \Delta\rho g = 6 \pi \mu v R$$

$$v_{\text{settling}} = \frac{2}{9\mu} R^2 \Delta\rho g$$

For a 1 micron ceramic particle ($\rho \approx 2500 \text{ kg} / \text{m}^3$) in water, the settling velocity

is about 1 micron per second.

(b) If turbulent, the settling velocity would be:

$$V \Delta\rho g = f_{\text{turbulent}} A K$$

$$(4/3) \pi R^3 \Delta\rho g = f (\pi R^2) (1/2 \rho v^2)$$

solve for v:

$$(4/3) R \Delta\rho g = f 1/2 \rho v^2$$

$$v = \frac{8 R \Delta\rho g}{3 \rho f}$$

Reynold's number:

$$\text{Re} = \frac{\rho v R}{2 \mu}$$

sub in our expression for v:

$$\text{Re} = \frac{4 \Delta\rho g R^2}{3 \mu f}$$

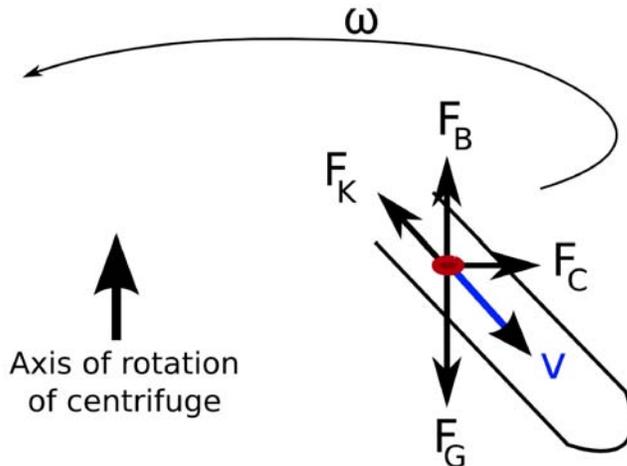
Turbulence is encouraged by:

1. Larger $\Delta\rho$ (i.e. denser particle)
2. Larger particle size (R)
3. Higher surface roughness (to increase f)

A lower fluid viscosity would also promote turbulence, but that isn't a property of the particle.

Problem 7

This is like problem 6, but with a 45 degree tilt, and the addition of the centrifugal force. The force diagram on a RBC looks like this:



We also know that an RBC at the top of the test tube must fall 10 cm in 1 minute, so $v = 0.001667$ m/s.

We will find the needed centrifugal force to settle the RBC:

$\sum F = 0$ along the direction of the velocity vector

$$F_{\text{Kinetic}} + F_{\text{Bouy}} = F_{\text{Gravity}} + F_{\text{Centrifugal}}$$

$$(6 \pi \mu v R) + \rho_{\text{solid}} V g \sin(\theta) = \rho_{\text{fluid}} V g \sin(\theta) + r_{\text{centrifuge}} \omega^2 \rho_{\text{solid}} \cos(\theta)$$

do some algebra, and substitute $V = (4/3) \pi R^3$, and $\rho = \rho_{\text{solid}}$, $\Delta\rho = (\rho_{\text{solid}} - \rho_{\text{fluid}})$:

$$\omega = \sqrt{\frac{4 \Delta\rho g \pi R^3 + 18 \sqrt{2} \mu \pi R v}{3 r_{\text{centrifuge}} \rho}}$$

plugging in values, we see that the centrifuge must have an angular velocity of at least $4.7 \cdot 10^{-6}$ rpm, meaning that we essentially need no centrifuging to separate RBCs from plasma.

Why, then, do centrifuges always spin really fast ($\sim 3000 - 4000$ rpm for 10 minutes) to separate blood cells from plasma? \Rightarrow Our assumptions were bad. The RBCs are NOT non-interacting, as they make up about 45% of the blood volume, so it can be very difficult for the plasma to flow around the RBCs when the viscosity is packed with other particles. Also, they are suspended in a polar fluid (\sim water), which has other consequences that we will discuss in coming lectures!

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