

# Pset 2 solutions

## 3.044 2013

### Problem I

Recall our old friend from pset 1, the solution to the heating/cooling of a 1D sheet with fixed temperature boundary conditions:

```
theta1D[ξ_, τ_, nmax_] :=  
  Sum[4 / (Pi n) Sin[n Pi ξ] Exp[-n2 Pi2 τ], {n, 1, 2 nmax - 1, 2}]
```

To produce the solution for a infinite square prism, we can use superposition, and  $\Theta(\xi, \psi, \tau) = \Theta(\xi, \tau) \Theta(\psi, \tau)$ . To get  $\Theta(\psi, \tau)$ , we can simply swap out  $\psi$  for  $\xi$  because the boundary conditions are the same in x and y. So,  $\Theta(\xi, \psi, \tau)$  is:

```
theta2D[ξ_, ψ_, τ_, nmax_] :=  
  Sum[4 / (Pi n) Sin[n Pi ξ] Exp[-n2 Pi2 τ], {n, 1, 2 nmax - 1, 2}]  
  Sum[4 / (Pi n) Sin[n Pi ψ] Exp[-n2 Pi2 τ], {n, 1, 2 nmax - 1, 2}]
```

To produce the solution for a cube, we can use superposition again:  $\Theta(\xi, \psi, \zeta, \tau) = \Theta(\xi, \tau) \Theta(\psi, \tau) \Theta(\zeta, \tau)$ .

```
theta3D[ξ_, ψ_, ζ_, τ_, nmax_] :=  
  Sum[4 / (Pi n) Sin[n Pi ξ] Exp[-n2 Pi2 τ], {n, 1, 2 nmax - 1, 2}]  
  Sum[4 / (Pi n) Sin[n Pi ψ] Exp[-n2 Pi2 τ], {n, 1, 2 nmax - 1, 2}]  
  Sum[4 / (Pi n) Sin[n Pi ζ] Exp[-n2 Pi2 τ], {n, 1, 2 nmax - 1, 2}]
```

Now to plot it, using contours, at various times. Assuming that we are cooling an initially hot body, hot colors = hot contours (pink is hottest) and cold colors = cold contours (purple is coldest) (flip the color scheme if we are heating an initially cold body):

First, define the desired contours, contour colors, dimensionless times, and how many terms to use in the series at each time:

```
contours = {0.01, 0.1, 0.25, 0.5, 0.75, 0.9, 0.99};  
colors = {Purple, Blue, Green, Yellow, Orange, Red, Pink};  
times = {0.001, 0.005, 0.01, 0.1, 0.2};  
terms = {30, 20, 10, 2, 2};
```

Then make a list of equations at each time:

```

plot1d = Table[ (theta1D[x, t, terms[[j]]] == contours[[i]]) /. t → times[[j]],
  {j, Length[times]}, {i, Length[contours]};

plot2d = Table[ (theta2D[x, y, t, terms[[j]]] == contours[[i]]) /. t → times[[j]],
  {j, Length[times]}, {i, Length[contours]};

plot3da =
  Table[ (theta3D[x, y, 0.25, t, terms[[j]]] == contours[[i]]) /. t → times[[j]],
    {j, Length[times]}, {i, Length[contours]};

plot3db =
  Table[ (theta3D[x, y, 0.5, t, terms[[j]]] == contours[[i]]) /. t → times[[j]],
    {j, Length[times]}, {i, Length[contours]};

plot3d = Table[ (theta3D[x, y, z, t, terms[[j]]] == contours[[i]]) /. t → times[[j]],
  {j, Length[times]}, {i, Length[contours]};

```

Create some labels:

```

fontsize = 28;
labelRow = {Text[Style["Time", fontsize]],
  Text[Style["1D", fontsize]], Text[Style["2D", fontsize]],
  Text[Style["3D, z = 0.25", fontsize]], Text[Style["3D, z = 0.5", fontsize]];
labelColumn = {Text[Style["τ = 0.001", fontsize]],
  Text[Style["τ = 0.005", fontsize]], Text[Style["τ = 0.01", fontsize]],
  Text[Style["τ = 0.1", fontsize]], Text[Style["τ = 0.2", fontsize]];

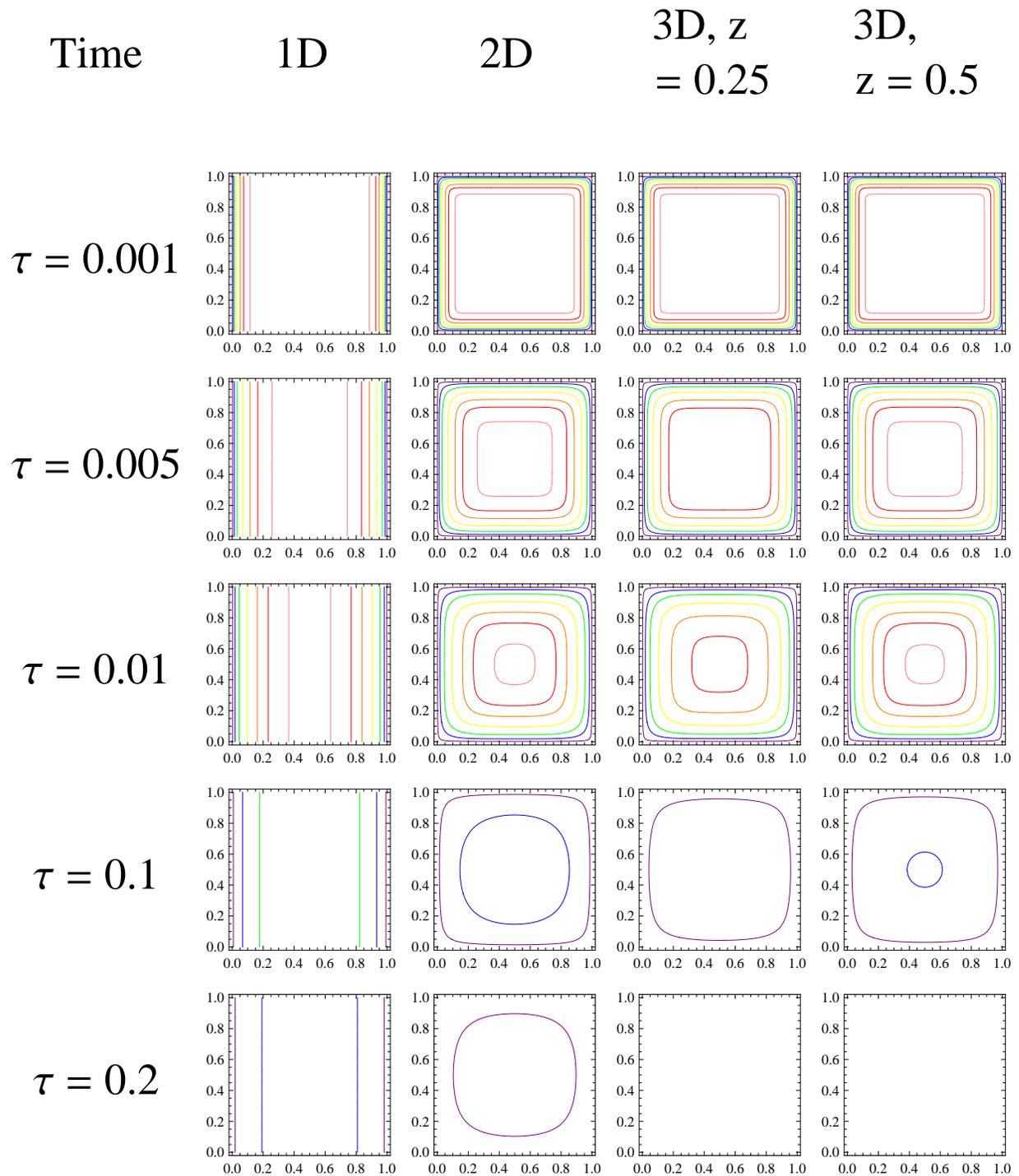
```

And put it all together in a graphics grid:

```

GraphicsGrid[Join[{labelRow}, Transpose[
  {labelColumn, Table[ContourPlot[Evaluate[plot1d[[j]]], {x, 0, 1}, {y, 0, 1},
    ContourStyle → colors, ContourLabels → True], {j, Length[times]}],
  Table[ContourPlot[Evaluate[plot2d[[j]]], {x, 0, 1}, {y, 0, 1},
    ContourStyle → colors, ContourLabels → True], {j, Length[times]}],
  Table[ContourPlot[Evaluate[plot3da[[j]]], {x, 0, 1}, {y, 0, 1},
    ContourStyle → colors, ContourLabels → True], {j, Length[times]}],
  Table[ContourPlot[Evaluate[plot3db[[j]]], {x, 0, 1}, {y, 0, 1},
    ContourStyle → colors, ContourLabels → True],
  {j, Length[times]}] ]]], ImageSize → Full]

```



Notice that as you add dimensions, the heat transfer goes faster. This makes sense: adding ways for heat to get in/out should accelerate the process. In 3D, that the center is a little behind the  $z = 0.25$  because it takes longer for heat to penetrate all the way to the center.

## Problem 2

**Problem:** Find the time required to heat a 25cm by 1m (long in the third dimension) steel billet to 900C at its center in a 1000C furnace.

**Geometry:** the center of the slab is at (0, 0).  $L_x = 0.125\text{m}$ ,  $L_y = 0.5\text{m}$ .

**BC's:** There are a total of 4 boundary conditions. Convection at  $x = L_x$  and  $y = L_y$ , symmetry ( $\frac{\partial T}{\partial x} = 0$  and  $\frac{\partial T}{\partial y} = 0$ ) at  $x = 0$  and  $y = 0$ .

**IC:** at  $t = 0$ ,  $T \sim 0\text{C}$  inside the billet, the furnace is at 1000C (just like we assumed in class)

**Governing equation:**

$$\text{Bi}_x = \frac{hL_x}{k} = \frac{(100 \text{ W/m}^2 \text{ K})(0.125 \text{ m})}{(35 \text{ W/mK})} = 0.36$$

$$\text{Bi}_y = \frac{hL_y}{k} = \frac{(100 \text{ W/m}^2 \text{ K})(0.5 \text{ m})}{(35 \text{ W/mK})} = 1.4$$

Therefore, we must use transient solutions from the charts for both dimensions, plus superposition.

$$\Theta(x, y, t) = \Theta(x, t) \Theta(y, t)$$

Use the Poirier & Geiger chart for a 1D plate,  $x/L = 0$ , for both dimensions.

**Solve:**

The goal is to find when  $\Theta(x, y, t) = 0.1$ . Take a guess that it will be less time than it was for the 1D case we analyzed in class, which was 22300 sec, so try 15000 sec:

$$\text{Fo}_x = \frac{\alpha t}{L_x^2} = \frac{(5.6 \times 10^{-6} \text{ m}^2/\text{s})(15000 \text{ s})}{(0.125 \text{ m})^2} = 5.4$$

$$\text{Fo}_y = \frac{\alpha t}{L_y^2} = \frac{(5.6 \times 10^{-6} \text{ m}^2/\text{s})(15000 \text{ s})}{(0.5 \text{ m})^2} = 0.34$$

read  $\Theta$ 's off the chart for the respective Biot numbers:

$$\Theta_x \approx 0.18$$

$$\Theta_y \approx 0.82$$

$$\Theta_{\text{center}} = \Theta_x \Theta_y = 0.15$$

This is still a little too cold, so try a later time of 18000 sec:

$$Fo_x = \frac{\alpha t}{L_x^2} = \frac{(5.6 \times 10^{-6} \text{ m}^2/\text{s})(18000 \text{ s})}{(0.125 \text{ m})^2} = 6.5$$

$$Fo_y = \frac{\alpha t}{L_y^2} = \frac{(5.6 \times 10^{-6} \text{ m}^2/\text{s})(18000 \text{ s})}{(0.5 \text{ m})^2} = 0.40$$

read  $\Theta$ 's off the chart for the respective Biot numbers:

$$\Theta_x \approx 0.14$$

$$\Theta_y \approx 0.77$$

$$\Theta_{\text{center}} = \Theta_x \Theta_y = 0.11$$

This is still just a little too cold, so try a later time of 19000 sec:

$$Fo_x = \frac{\alpha t}{L_x^2} = \frac{(5.6 \times 10^{-6} \text{ m}^2/\text{s})(19000 \text{ s})}{(0.125 \text{ m})^2} = 6.8$$

$$Fo_y = \frac{\alpha t}{L_y^2} = \frac{(5.6 \times 10^{-6} \text{ m}^2/\text{s})(19000 \text{ s})}{(0.5 \text{ m})^2} = 0.43$$

read  $\Theta$ 's off the chart for the respective Biot numbers:

$$\Theta_x \approx 0.13$$

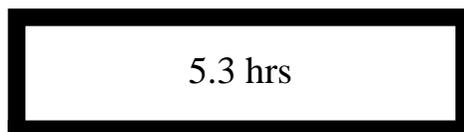
$$\Theta_y \approx 0.75$$

$$\Theta_{\text{center}} = \Theta_x \Theta_y = 0.10$$

*This time is equivalent to 5.3 hours.*

In class, we learned that the time to heat a 25cm square billet (long in the third dimension) is about 3.1 hours, and the time to heat a 25cm thick billet (long in the other two dimensions) is about 6.2 hours. Our analysis showed that the time to heat a 25cm x 100cm billet (long in the third dimension) is in between these values, at 5.3 hours.

Time to reach 900°C:



These times make sense because the contribution in heat from the sides decreases as the width of the billet increases.

## Problem 3

We want a solution for a 1D slab, with uniform heat generation within, and convection on the surfaces.

Carslaw & Jaeger, Chapter on linear heat flow in a solid bounded by two planes, section 3.14: the slab with heat produced within it, equation 12, p. 132:

$$T = \frac{1}{2kL} \left\{ (x+L) \int_x^L d\eta \int_{-L}^{\eta} A(\xi) d\xi + (x-L) \int_{-L}^x d\eta \int_{-L}^{\eta} A(\xi) d\xi \right\}$$

where  $T$  = temperature,  $k$  = thermal conductivity,  $A(x)$  is the rate of heat production (which we will take to be a constant in our case that only depends on the current), and  $L$  = sheet thickness.  $\eta$  and  $\xi$  are “dummy” variables for integration, so they effectively go away and don’t mean anything. You are not required to do the next bit, however, it is cool:

In our case,  $A(x) = A = \text{constant}$  rate of heat production, so

$$\int_{-L}^x d\eta \int_{-L}^{\eta} A(\xi) d\xi = A \int_{-L}^x (\eta + L) d\eta = A \left( \eta^2 / 2 + L\eta \right) \Big|_{\eta=-L}^{\eta=x} = A \left( x^2 / 2 + Lx + L^2 / 2 \right)$$

and

$$\int_x^L d\eta \int_{-L}^{\eta} A(\xi) d\xi = A \int_x^L (\eta + L) d\eta = A \left( \eta^2 / 2 + L\eta \right) \Big|_{\eta=x}^{\eta=L} = A \left( -x^2 / 2 - Lx + 3L^2 / 2 \right).$$

Therefore,

$$\begin{aligned} T &= \frac{1}{2kL} \left\{ (x+L) A \left( -x^2 / 2 - Lx + 3L^2 / 2 \right) + (x-L) A \left( x^2 / 2 + Lx + L^2 / 2 \right) \right\} \\ &= \frac{A}{2kL} \left\{ L^3 - Lx^2 \right\} \\ &= \frac{A}{2k} \left\{ L^2 - x^2 \right\} \end{aligned}$$

where  $A$  is the rate of heat generation, so it is just a parabola! This is a solution that you can do by hand, it turns out.

## Problem 4

We want a solution for a single wall, so that is a 1D slab, with a sinusoidally varying temperature at one surface.

C & J, Chapter on linear heat flow in a solid bounded by two parallel planes, Section 3.6: the slab with periodic surface temperature, equation 5, page 105:

$$T = A \sin(\omega t + \epsilon + \phi) + 2\pi\kappa \sum_{n=1}^{\infty} \frac{n(-1)^n (\kappa n^2 \pi^2 \sin \epsilon - \omega L^2 \cos \epsilon)}{\kappa^2 n^4 \pi^4 + \omega^2 L^2} \sin \frac{n\pi x}{L} e^{-\kappa n^2 \pi^2 t / L^2} + T_{\text{inside}}$$

Where  $T$  = temperature,  $L$  = wall thickness, the temperature is kept at  $T_{\text{inside}}$  at

$$x=0, A = \left\{ \frac{\cosh 2kx + \cos 2kx}{\cosh 2kl + \cos 2kl} \right\}^{1/2}, \phi = \arg \left\{ \frac{\cosh kx(1+i)}{\cosh kl(1+i)} \right\}, \mathbf{k} = \left( \frac{\omega}{2\kappa} \right)^2, \epsilon = \text{phase shift}$$

(which we may take to be zero),  $\omega$  = frequency of temperature variation (which we take to be 1 day), and  $\kappa$  = thermal diffusivity. Note that this is essentially what you'd expect: if the surface temperature is sinusoidal, the solution is sinusoidal in space, but exponentially damped in time, i.e., the effects of the outside temperature are exponentially damped as you move into the thickness of the wall.

## Problem 5

We want a solution for a solid sphere, with uniform heat generation, and a radiative boundary condition at the surface, because the antenna can cool by radiating into the vacuum of space, which is at  $\sim 0$  K. There is no convection in space, so our answer better not have a convective heat transfer coefficient in it.

C & J, chapter on the flow of heat in a sphere and cone, section 9.8: the sphere  $0 \leq r < a$  with heat generation, equation 19, page 245:

$$T = \frac{A_0}{6hk} \{h(a^2 - r^2) + 2a\} - \frac{2ha^2 A_0}{rk} \sum_{n=1}^{\infty} \frac{\sin r \alpha_n e^{-\kappa \alpha_n^2 t}}{\alpha_n^2 [a^2 \alpha_n^2 + ah(a h - 1)] \sin a \alpha_n}$$

where  $T$  = temperature,  $A_0$  = constant heat generation,  $h$  =  $M$  number divided by the characteristic length so that it has units of  $1/m$  (note: this is really hard to find! It's kind of explained on page 19, but it takes some digging and thinking),  $k$  = thermal conductivity,  $\kappa$  = thermal diffusivity,  $a$  = radius of the antenna,  $r$  = radial coordinate, and  $\alpha_n$  are the roots of

$$a \alpha \cot a \alpha = 1 - ah$$

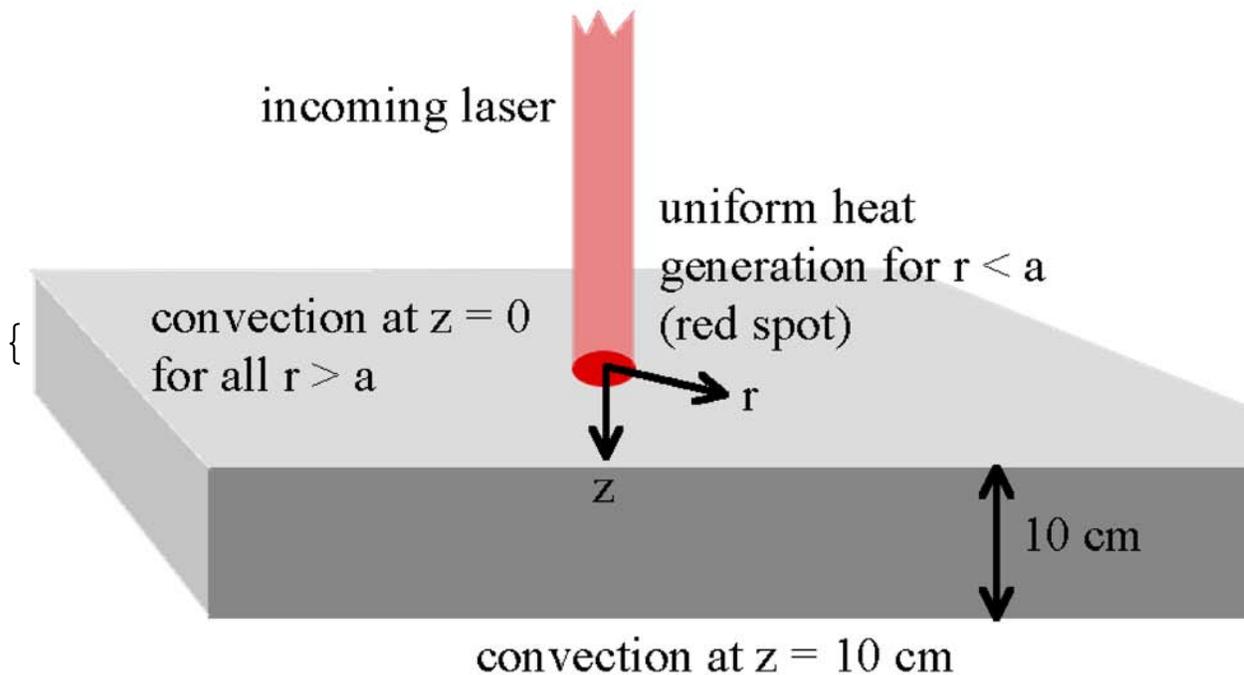
for  $n = 1, 2, \dots$

## Problem 6

### i. Geometry & BCs

-Center of laser spot is origin

- laser spot has radius  $a$
- symmetry at  $r = 0$ :  $\partial T / \partial r = 0$ .



## ii. Quantitative simplification

Calculation the Biot number for the  $z$  direction:

$$Bi = h L / 2 k = (10 \text{ W} / \text{m}^2\text{K}) (0.05\text{m}) / (35 \text{ W/mK}) = 0.014$$

The Biot number is small  $\rightarrow$  we can neglect heat transfer via convection, all heat transfer will occur in the interior of the steel slab.

Therefore, look for a solution  $T(r, z)$ . Change the difficult convective boundary condition to zero flux at the  $z=0, r>a$  surface.

Note: some folks might be confused why our solution will still depend on  $z$ , even though the Biot number is small in  $z$ . This is because the Biot number compares conduction in  $z$  with convection at the surface. Convection is so slow that it won't contribute at all to the transfer of heat. However, conduction in  $z$  occurs at the exact same speed as conduction in  $r$  because the slab is isotropic. So, if our solution depends on  $r$ , it better depend on  $z$  too, because conduction is equally fast in  $r$  and  $z$ . All that we can conclude from a small Biot number here is that we

can neglect heat loss to the air, all heat will be dissipated internally.

### iii. Find the steady-state solution, plot it, and identify melt (if any)

What is the solution going to look like? Many people expect that you should treat the laser as a point source and treat the body as finite in  $z$ . But, the thing that sets the relevant length scale in any heat transfer problem is not the dimensions of the objects involved, but the dimensions of the heat transfer process. The heat is coming in over a little disk of radius 0.1mm and spreading radially outward in  $r$  and  $z$ . It doesn't care about the surface at  $z = 10\text{cm}$  because the length scale of the actual heat transfer is orders of magnitude smaller. As far as the heat is concerned, that surface is too far away to matter, so you can treat it a semi-infinite in  $z$ .

This relevant solution can be found in Carslaw & Jaeger, the chapter on the flow of heat in regions bounded by surfaces of the cylindrical coordinate system, section 8.2: the steady temperature in an infinite or semi-infinite medium due to heat supply over a circular area, page 216, equation 14, "the region  $z > 0$  with constant flux  $Q$  over the circular area  $r < a$  and zero flux over  $r > a$ ":

$$T = \frac{Qa}{k} \int_0^\infty e^{-\lambda z} J_0(\lambda r) J_1(\lambda a) / \lambda d\lambda$$

where  $T$  = temperature,  $Q$  = heat flux over the circle  $0 < r < a$ ,  $k$  = thermal conductivity,  $\lambda$  = dummy variable,  $z$  = coordinate into the slab,  $r$  = radial coordinate,  $J_0$ ,  $J_1$  = zeroth and first kind Bessel J functions. Now, plot this result:

The heat flux is equal to the wattage of the laser divided by the laser spot area to yield  $W/m^2$ :

$$Q = 1000 / (0.0001^2 \text{ Pi})$$

$$3.1831 \times 10^{10}$$

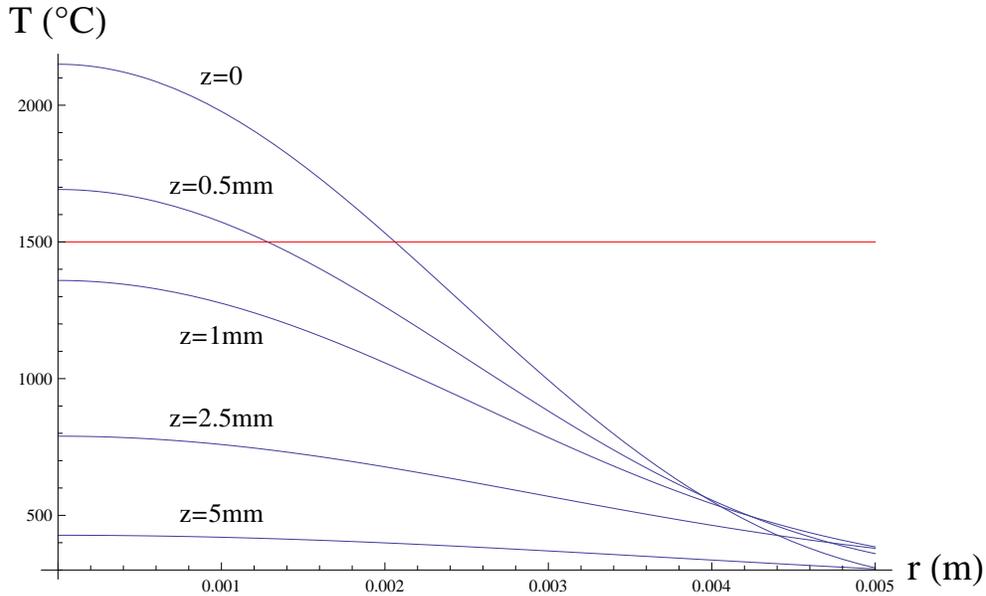
Create a function  $T(r, z)$  using our solution:

```
temperature[r_, z_] := Q 0.0001 / (2 * 37)
  NIntegrate[Exp[-1 z] BesselJ[0, 1 r] BesselJ[1, 1 0.0001] 1 / 1, {1, 0, 1000}]
```

Plot  $T(r, z)$ :

```
trz = Plot[Table[temperature[r, z], {z, {0, 0.0005, 0.001, 0.0025, 0.005}}],
  {r, 0, 0.005}, AxesLabel -> {Style["r (m)", 20], Style["T (°C)", 20]}];
```

```
Show[trz, Plot[1500, {r, 0, 0.005}, PlotStyle -> Red],
Graphics[{Black, Text[Style["z=0", 14], {0.001, 2100}],
Text[Style["z=0.5mm", 14], {0.001, 1700}],
Text[Style["z=1mm", 14], {0.001, 1150}], Text[Style["z=2.5mm", 14],
{0.001, 850}], Text[Style["z=5mm", 14], {0.001, 500}]}]]
```



In the above plot, we see that the temperature exceeds the melting point, 1500C, for about 0.75 mm into the slab in  $z$  and for a radius out to about 2.2 mm at the surface (ignoring the latent heat, which would make the melt puddle even smaller). There is no way this laser could cut this slab of steel.

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