3.032 Problem Set 7 Solutions

Fall 2007

Due: **Start** of Lecture, Friday 12.07.07

1. If one's appendix becomes infected with bacteria, it can rupture or perforate. The contents of the infected appendix then leak into the abdomen, leading to periappendiceal abscess (a collection of infected pus in the abdomen and pelvis), which is as bad as it sounds and can be fatal.

Although perforation of the appendix requires immediate treatment whether fast or slow, a leak-before-break condition is preferable because this gives more time for the patient to have the slowly leaking appendix removed via surgery.

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Fig. 1: Human appendix (uninfected) is a cylindrical appendage extending from the cecum of the large intestine. The typical size of an appendix is 10 cm long x 1 cm in diameter, with a wall thickness of about 1 mm. At rupture, the internal pressure from the infection reaches about 1 MPa.

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(a) Idealize the appendix wall material as an isotropic, elastic-to-brittle solid, and determine the critical crack size *a* of the appendix wall required for the appendix to leak-before-breaking. Let 2a = 2c, assuming a semicircular through-thickness "crack" in the appendix wall.

Solution: If a > t, the structure will leak-before-break (LBB).

If the wall t = 1 mm, aLBB = 1mm since a = c in this half-penny crack.

As a check, I'll find ac from KIc = fs sqrt(pac), and assume f = 1 for simplicity.

$a_c = 1/\pi(K_{Ic}/\sigma)^2$

σ *from p = 1 MPa* $\rightarrow \sigma_h = pr/t = 5$ *MPa*

Kic from literature for soft tissues = 0.1 MPa sqrt(m) [Source: Ashby map]

Thus, ac = 0.13 mm. Since a_c *from Kic is less than* a_{LBB}*, we know that the wall will fracture catastrophically at* $a = a_c$ *long before a reaches t (LBB). Thus, although* $a_{LBB} = 1$ *mm, it is not attainable prior to catastrophic fracture (according to this model).*

(b) Comment on whether this prediction is reasonable, vis a vis the size of the appendix and the relative infrequency of ruptured appendices.

Solution: Not reasonable, since appendices do not rupture all the time. Likely that energy dissipation at "crack tips" via viscoelastoplasticity occurs, and also likely that Kic (plane strain fracture toughness) poorly captures the actual Ki of this shin sheet of appendix wall material (closer to a state of plane stress, as were all our thin-walled pressure vessel problems).

As we know, Ki is higher in plane stress than in plane strain, which is why material samples must be very thick to accurately measure KIC that is independent of sample dimensions.

(c) Explain how you would determine the critical wall thickness *t* of other organ "pressure vessels" such as the bladder, if you knew a pre-existing crack size (say, from a surgical incision) and needed to determine the critical thickness of the organ wall for a specific magnitude of internal organ pressure *p*.

Solution: If I knew s = pr/t and an initial "a" was given, tcritical is for tcrit < acrit.

As long as a < tc, $a_c = 1/\pi(K_{Ic}/\sigma)^2$.

Then Kic = pr/t sqrt(πa_c*)* → $t_{crit} = pr(\pi a_c)/Kic$*, where we assume* a_c *is the incision size.*

(d) Assuming appendix rupture really was well described by brittle fracture. Prof. X's appendix burst at a critical crack length *a* = 0.1 mm, under an internal pressure that was 5 MPa just prior to catastrophic failure. What were the continuum mechanical properties K_{IC}, G_{IC}, and J_{IC} of Prof. X's appendix wall material, which is mostly smooth muscle?

Solution: ac = 0.1 mm, and p = 5 MPa → σ_h *= pr/t = 25 MPa.*

Kic = 25 MPa sqrt(p(0.1 mm)) = 0.44 MPa sqrt(m)

Gic = $Kic^2(1 - \nu^2)/E$ *in plane strain =* $[0.44\ MPa\ sqrt(m)]^2(1 - 0.45^2/10\ MPa)$
= 0.015 MPa m = 0.015 MJ/m

Here, I assumed E = 10 MPa, a typical soft tissue E from the literature.

(e) Why is the appendix poorly described by Griffith's fracture criterion?

Solution: Appendix material is protein/cell based and elastomeric → nonlinear elastic, and not brittle. Griffith is for linear elastic materials that are brittle upon failure.

2. The stresses around a crack tip are "magnified" because the crack faces are displaced a distance **u** inside the material, creating a strain $\boldsymbol{\varepsilon}(r, \theta, a)$ and thus a stress $\boldsymbol{\sigma}(r, \theta, a)$ inside the material. This is analogous to the stresses created by a dislocation inside a material, though the symmetry breaking is different

The stresses around the crack tip under plane strain conditions are given by
$\sigma_{xx} = \{K_I/[2\pi r]^{1/2} \cos\theta/2\}\ (1 - \sin[\theta/2]\ \sin[3\theta/2])$

$\sigma_{yy} = \{K_I/[2\pi r]^{1/2} \cos\theta/2\} (1 + \sin[\theta/2] \sin[3\theta/2])$
$\sigma_{xy} = \{K_I/[2\pi r]^{1/2} \cos\theta/2\} (\sin[\theta/2] \cos[3\theta/2])$

(a) What does "plane strain" mean in terms of the dimensions of the material that contains the crack and the way in which the crack is loaded?

Solution: Plane strain means that the piece is so thick in the through-crack thickness direction (into the page) that the strain in the z (if z is the Mode I crack opening loading direction) is zero (no displacement in that direction). All strain is in the x-y or crack plane.

(b) Determine the other normal stress σ_{zz} and the shear stresses σ_{xz} and σ_{yz} in terms of these stresses and any other required mechanical properties of the material, remembering that linear elastic fracture mechanics idealizes the material as an isotropic elastic continuum.

Solution: $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = \nu[Ki/sqrt(2\pi r) \cos \theta/2]$

And shear stresses s_{xz}, $s_{yz} = 0$.

(c) Graph the largest of these stress components as a function of distance from the crack tip. Here, you can normalize by any quantities you do not know, such as the magnitude of applied stress.

*Solution: Let q = 0, and then sxx = syy = Ki/sqrt(2pr)*1; sxy = 0; szz = 0.45*2sxx.*

Can plot these on normalize axes as $s_{xx}/Ki/sqrt(2\pi)$ *vs. r, where r is distance from the crack tip in the x direction. This stress decays with distance from the crack tip as sqrt(1/r).*

(d) The radius of the plastic zone around a crack tip r_p is given by the distance from the crack tip over which the stress exceeds the yield stress of the material. Determine the size of this plastic zone $r_p(\sigma, a, \sigma_y)$ by evaluating the crack tip stresses σ_{ij} at $\theta = 0$.

Solution: For the size of the plastic zone, rp, by definition the stress must exceed the material yield strength:

sxx (or syy) = s_y*, the material yield strength.*

Thus, $Ki/sqrt(2\pi r) > s_y$ *defines the extent of rp. Solving this equality for rp,* $rp \leq 1/2\pi * (Ki/ s_y)^2$.

(e) Now compare the size of this plastic zone for a crack of length a = 1 mm under a Mode I stress σ = 100 MPa in Au, Cu, W, Si, and amorphous SiO_2.

Solution: With this definition of rp in (d) and the calculation of $Ki = s\ sqrt(\pi a)$ *= 5.6 MPa sqrt(m) for s = 100 MPa and a = 1 mm, we find:*

Matl	YS (MPa)	rp (m)
Au	100	4.93E-03

Cu	33	4.52E-02
W	750	8.76E-05
Si	120	3.42E-03
silica	20000	1.23E-07

So, clearly, ductile metals generally have large rp, but Si which is brittle has a plastic zone that is theoretically on the order of Au and Cu. The difference between these metals and this semiconductor is that dislocation motion requires much more energy/stress in Si than in Au or Cu, so energy that cannot be dissipated by sustained plasticity is dissipated by fracture instead.

3. From the literature, determine the Young's elastic modulus *E*, yield strength σ_y and fracture toughness K_{IC} of any three materials of interest to you.

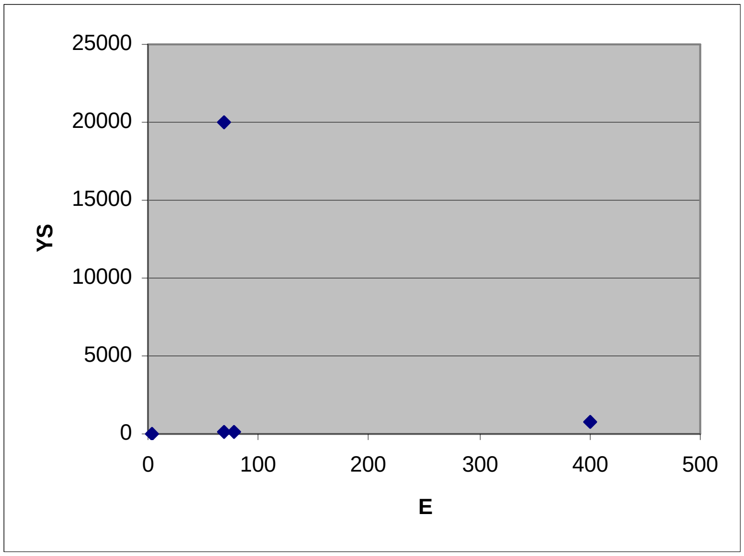
(a) Graph σ_y vs. K_{IC}, E vs. σ_y, and E vs. K_{IC}. Comment on the observed trends (noting that there may be no clear trend in some cases).

Solution:

Material	E [Gpa]	YS [MPa]	K_IC [MPa rt m]
Au	77	100	18
W	400	750	70
polystyrene	3	1	1
Al	68	100	20
silica glass	68	20000	0.7

There may be some apparent trends, but these are coincidences due only to the choice of materials included in your analysis. For example, as we've discussed several times, Al and silica glass have comparable elastic moduli but very different plastic properties and fracture properties. The origin of elastic properties is resistance to bond stretching; the origin of plastic properties is resistance to dislocation nucleation/motion/multiplication (in crystals) or shear banding/crazing (in amorphous metals/polymers, respectively); the origin of fracture properties is resistance to bond breaking on a macroscale (lots of bonds). Although within a given material class there may be some trends (e.g, the stronger a material, the less ductile or more brittle it is; or, the higher the yield strength, the higher the fracture stress), these are not universal trends because the molecular determinants of each of these three kinds of mechanical properties are different.

In the graphs below, the inclusion of glass makes it clear there are NO trends apparent, except that yield strength and fracture toughness increased together in this sampling of materials.



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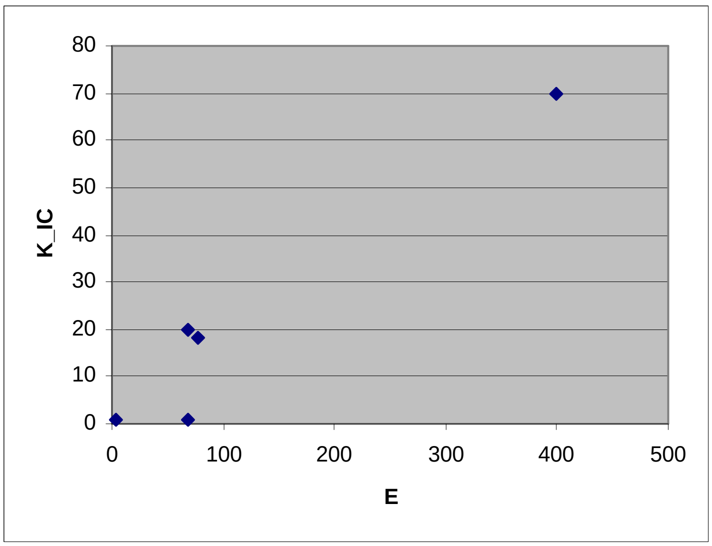

80
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K_IC
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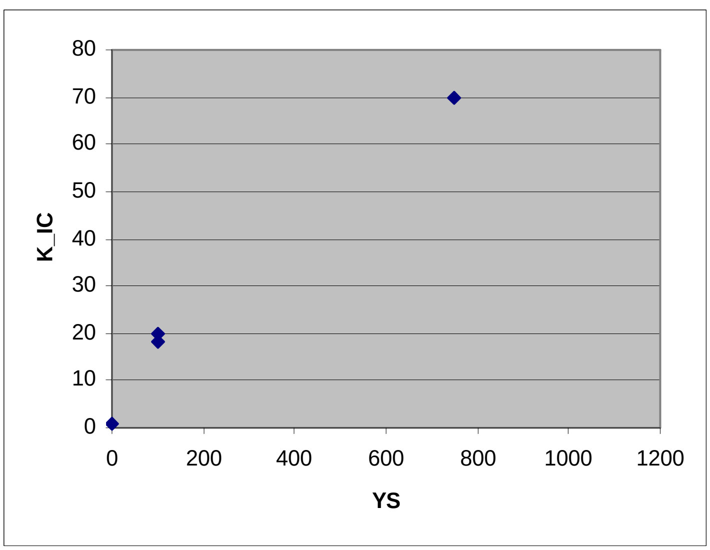

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K_IC
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200
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600
800
1000
1200
YS

(b) Given these trends, how would you design a general microstructure for which the application demanded that the material be stiff, strong, and tough. Be as specific as possible, and feel free to draw this schematic microstructure to illustrate your reasoning.

Solution: To increase the stiffness, I would choose a material composition (element) that had high stiffness (e.g., W). To increase the strength, I would use any of our strengthening mechanisms in this crystalline metal, e.g., grain size reduction or solute strengthening. To increase the fracture toughness, I would then decide not to introduce solutes (which could tend to segregate to the high energy grain boundaries), and would instead choose grain size reduction and make sure that the grain boundaries were free of any impurities that might lead to intergranular fracture. I'd also polish the surface of that material to reduce number/size of pre-existing defects that act as stress concentrations.

(c) Figure 2 shows a scanning electron micrograph of a fracture surface. Is this of a metal, polymer, or ceramic, and is it indicative of ductile fracture, brittle intergranular fracture, or brittle transgranular fracture?

Solution: Transgranular brittle fracture, since no gb features are apparent, as typical in intergranular fracture, and the surface is not rough and cotton-candy like, as typical in ductile fracture.

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Fig. 2: SEM fractograph of material. Scalebar= 100 um.

Fig. 3: Fatigue striations evident in SEM micrograph of 302 stainless steel spring that has fractured.

Image removed due to copyright restrictions. Please see: any fractograph of 302 stainless steel, such as Fig. 6b in Schuster, G., and Altstetter, C. "Fatigue of Annealed and Cold Worked Stable and Unstable Stainless Steels." *Metallurgical Transactions* 14A (October 1983): 2077-2084.

4. Figure 3 shows the fracture surface of a 302 stainless steel spring. This spring was under a cyclic stress between 0 and 100 MPa at a frequency of 1 kHz. We can assume that the initial crack size *a* was at the limit of the resolution of an optical microscope, with which the spring

was inspected before use. Young's elastic modulus E, yield strength σ_y and fracture toughness K_{IC} of this steel are 210 GPa, 500 MPa, and 100 MPa $m^{1/2}$, respectively.

(a) Calculate the crack growth rate during steady-state crack propagation, da/dN. Compare this with the average da/dN you measure from the fractograph in Fig. 2b.

Solution:
Stress amplitude = 100 MPa; frequency = 1000 Hz; a ~ 400 nm (wavelength of visible light, though Rayleigh criterion actually says resolution of optical images is ~$\lambda/2$).

$da/dN = C(\Delta K)^m$

From the literature for stainless steel (extruded wire used for springs), C = 1 x 10^{-9} – 5 x 10^{-10} (units of which depend on corresponding m) and m = 2.94 – 3.88 (but I'll assume 2).
Source: Googled "stainless steel Paris law constants" and found 2006 paper: Sriharsha, HK et al. Eng. Fract Mech 64 (1999), 607. Towards standardizing a sub-size specimen for fatigue crack propagation behavior of nuclear pressure vessel steel".

Table 3
Paris law constants for TPB and CT specimens

Specimen type	Dimensions (mm)	C	m
Three point bend	$W = 10, B = 10, S = 40$	3.84×10^{-9}	2.94
Three point bend	$W = 20, B = 10, S = 40$	1.12×10^{-9}	3.40
Three point bend	$W = 20, B = 10, S = 80$	5.15×10^{-10}	3.64
Compact tension	$W = 50, B = 10$	2.52×10^{-10}	3.88
Compact tension	$W = 50, B = 20$	6.84×10^{-8}	2.33
ASME		4.77×10^{-10}	3.726

Using C = 4 x 10-9 and m = 2, da/dN = 6 x 10^{-12} m/cycle, or less than 1 Angstrom per cycle.

Looking at the fractograph, da/dN on average is about 1.3 um/cycle. Thus, either my C and m are inaccurate or the initial crack length was NOT a = 400 nm.

(b) Assuming the crack was already at the critical crack length to propagate at this applied stress, how many minutes was the spring in use before fatigue failure? Note that failure time is a product of the number of cycles to failure and the cyclic operating frequency of the structure.

Solution: From equations used in previous problems, critical crack length ac = 0.32 m = 320 mm for this level of stress and value of Kic.

If the crack were already THIS large, fracture would have been instantaneous. Here, "critical crack length to propagate" does NOT means a = ac, but a propagating at da/dN greater than Angstroms/cycle (above ΔK threshold on da/dN vs. ΔK graph).

I'll find N from ai = 400 nm (assuming it is propagating at this size, but slowly) to af = ac = 320 mm.

Then, $N_f = 1/[C\Delta\sigma^2\pi]*\ln(a_f/a_i)$ *for m = 2 and assumed f = 1.*

Thus, Nf = 108,219 cycles and tf = Nf/frequency = 108 seconds!

This is a very short failure time for a spring, especially considering one that had invisible defects at the start of the Paris law regime. Likely that C and m values are inaccurate.

To compare, the fractograph shows that in AT LEAST THIS SMALL REGION OF THE FATIGUE CRACKING SAMPLE, the crack is propagating at 10^{-6} *m/cycle * 1000 cycles/sec = mm/sec. If the spring were a few mm in diameter, this would give about the same answer, but the spring diameter is not given in the problem.*