Fourier Series: Decomposition into periodic functions.

I. Defining projection in function space, one way is as an integral over a domain.

$$\begin{matrix}\vec{a}\cdot\vec{b} \\ \langle a|b\rangle\end{matrix} \rightarrow \int_D a(x)^* \cdot b(x)dx = \langle a|b\rangle$$

$D\colon -\infty < x < \infty$ General functions

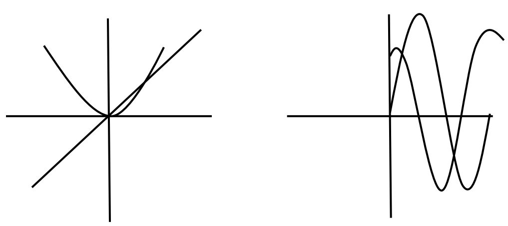
$D\colon -\pi < x < \pi$ Periodic functions

$D\colon -p < x < q$ General restricted domain

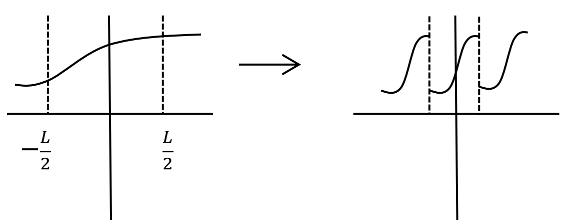
The projection is only valid over the domain you integrate

Normalized function: $\langle a|a\rangle = |a|^2 = 1$

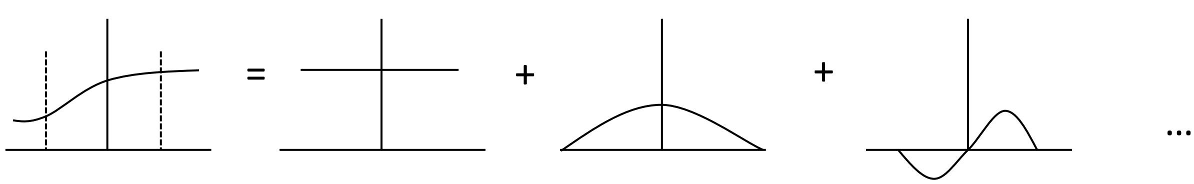
Orthogonal functions: $\langle a|b\rangle = 0$



II. Periodic functions: Forier Series, as some a portion of a periodic or aperiodic function is periodic.



Now break that portion into a sum of periodic functions.



Why can we do this (easily)?

$a_{n(x)} = \frac{2}{L}\cos^2\frac{n\pi}{L}x \qquad b_{m(x)} = \frac{2}{L}\sin\frac{2m\pi}{L}x \qquad a_{0(x)} = \frac{1}{L} \qquad n, m = 1,2,3 \ldots$

$D = -\frac{L}{2} \cdot\cdot \frac{L}{2} \qquad \langle a_i | b_j \rangle = 0 \qquad \langle a_i | a_j \rangle = 0 \qquad \langle a_i | a_i \rangle = 1$

Orthonormal basis! (Maybe of some differential eq…)

Another way to express:

Euler's equation: $e^{i\theta} = \cos\theta + i\sin\theta \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{i\theta}}{2}$

Normalizing

$$\int_{-\pi}^{\pi} (e^{i\theta} \cdot e^{-i\theta}) d\theta = 2\pi \to L$$

$c_p(X) = \frac{1}{L} e^{i\frac{2\pi p}{L}x} \qquad p = 0, \pm 1, \pm 2, \pm 3 \ldots$

III. Fourier Series Proper

$$f(x)_L = \alpha_0 + \sum_{i=1..\infty} \alpha_i \cos\frac{2\pi i}{L}x + \beta_i \sin\frac{2\pi i}{L} = \sum_{j=-\infty..\infty} \gamma_j e^{i\frac{2\pi j}{L}x}$$

$$\propto_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) dx$$

$$\propto_i = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{f(x)} \cos\frac{2\pi i}{L} x dx$$

$$\beta_i = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{f(x)} \sin\frac{2\pi i}{L} x dx$$

$$\gamma_j = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{f(x)} e^{-i\frac{2\pi j}{L}X} dx$$

Note that the complex form and the sine/cosine form are equivalent as for each value of *i*, the sine is a difference and the cosine is a sum of two exponentials. We like using the sines and cosines because they are real functions while the exponential ones are complex and have complex coefficients. If you plug a real function into the complex Fourier series, some sum of sines and cosines will pop out at the end.

Example

F=x L=1

$$\gamma_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x\, dx = 0$$

$$\gamma_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{i2\pi x}\, dx = \frac{1}{2\pi i}$$

$$\gamma_{-1} = \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{-i2\pi x}\, dx = -\frac{1}{2\pi i}$$

$$\gamma_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{i4\pi x}\, dx = \frac{1}{4\pi i}$$

$$\gamma_{-2} = -\frac{1}{4\pi i}$$

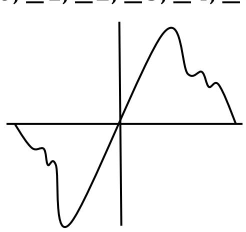
$$\gamma_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{i2\pi n x}\, dx = \frac{1}{2\pi n i}$$

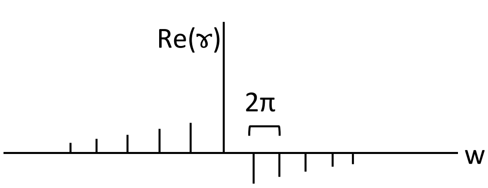
n=±1..∞

$$f(x)_L = \sum_{n=1\ldots\infty} \frac{1}{n\pi}\left(\frac{e^{i2\pi nx} - e^{-i2\pi nx}}{2i}\right) = \sum_{n=1\ldots\infty} \frac{\sin 2\pi nx}{\pi n}$$

Since x is real and odd, our complex series resulted in a sum of sines with real coefficients.

$0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$





$$w = \frac{2\pi n}{L}$$

IV. Fourier transform

What happens to our coefficient plot as we increase L?

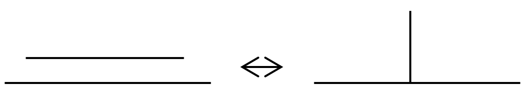
The spaces get smaller and smaller until…

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx \rightarrow F(f(x))$$

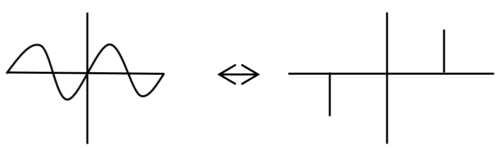
Now the coefficients are a continuous variable that tell us about the frequency breakdown of a given function.

Let's look at some examples:

The constant function doesn't oscillate at all, so is just a delta function at the origin, by converse a sharp pulse (delta function in position), has all of the frequencies.



A sine or cosine, due to Euler's formula, are delta functions at plus/minus the frequency



In general the wider a pulse is in real space, the sharper it will be in frequency space