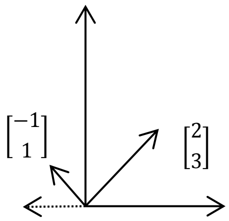
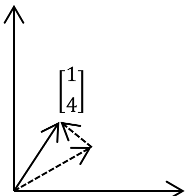
I. Vectors, Vector Addition, Vector Notations

Vectors



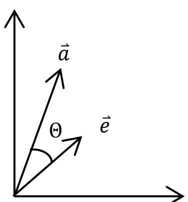
Vector Addition



Some Vector Notations

Vector	Matrix	Unit Vector
$\vec{e}$	M	$\hat{e}$
e_i	M_{ij}	$\hat{e}_i$
$\vert e\rangle$	$\widehat{M}$	$\vert e\rangle$

II. Vector "Multiplication"



$\vec{e} = \begin{bmatrix}2\\3\end{bmatrix}$ $\vec{a} = \begin{bmatrix}1\\5\end{bmatrix}$

Dot product is as close to multiplication as vectors have

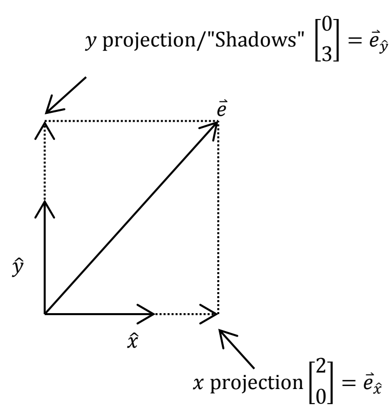
$$\vec{e}\cdot\vec{a} = \vec{e}^T\vec{a} = [2\ 3]\begin{bmatrix}1\\5\end{bmatrix} = 2\cdot 1 + 3\cdot 5 = 17 = |\vec{e}||\vec{a}|\cos\Theta$$

$$\vec{e}\cdot\vec{e} = |\vec{e}||\vec{e}|\cos 0 = |\vec{e}|^2 \rightarrow= [2\ 3]\begin{bmatrix}2\\3\end{bmatrix} = 13$$

Normalization: Dot product of something with itself is equivalent to its length/magnitude

Unit Vector $\hat{e} = \frac{1}{\sqrt{13}}\begin{bmatrix}2\\3\end{bmatrix} = \frac{\vec{e}}{|\vec{e}|}$

III. Projection interpretation of dot product



$\hat{x} = \begin{bmatrix}1\\0\end{bmatrix}$ $\hat{y} = \begin{bmatrix}0\\1\end{bmatrix}$

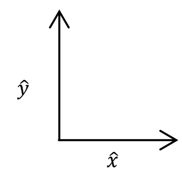
$$\vec{e}\cdot\hat{x} = [2\ 3]\begin{bmatrix}1\\0\end{bmatrix} = 2 = |\vec{e}_{\hat{x}}|$$

$$\vec{e}\cdot\hat{y} = [2\ 3]\begin{bmatrix}0\\1\end{bmatrix} = 3 = |\vec{e}_{\hat{y}}|$$

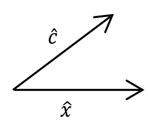
The dot product may be thought of as how much one vector and another are related.

IV. Basis

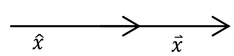
$\hat{x}$ and $\hat{y}$ are orthogonal or normal basis that are complete i.e. can map any vector in 2D.



Descartes' basis is complete, but not orthogonal.



$\hat{x}$ and $\vec{x}$ are neither normal or complete



V. Matrixes are operations

Identity: returns any vector multiplied by it (the “1” of the vector-space)

$$I\vec{a} = \vec{a} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

x Stretch: doubles the x-value of any vector

$$S\vec{a} = \begin{bmatrix} 2_{a_1} \\ a_2 \end{bmatrix} \quad S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotation: Rotates any vector about the origin by angle Θ

$$R = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix}$$

Projection: A very important matrix, gives the basis vector weighted by the projection of the vector its applied on

$$P_{\hat{a}}\vec{e} = (\hat{a} \cdot \vec{e})\hat{a} = \hat{a}\hat{a}^T\vec{e}$$

$$P_{\hat{a}} = \hat{a}\hat{a}^T$$

VI. Eigenvalues, Eigenvectors

$M\vec{e} = \varepsilon\vec{e}$

If $\vec{e}$ is an eigenvector of M, multiplying $M\vec{e}$ is the same as multiplying $\varepsilon\vec{e}$, where ε is the constant eigenvalue of the eigenvector.

An n x n matrix can have no more than n eigenvalues. If it has n non-zero values, then it has a complete eigenbasis.

For example all vectors are eigenvalues of the identity matrix. This is because the I matrix has n eigenvalues that are all 1, so any n distinct, independent vectors could be its eigenbasis. For conveniences, we choose an orthogonal basis whenever possible.

Let's try a different matrix.

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

To find the values and vectors we introduce the determinant

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

$$det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} + & - & + \\ a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a(ei - fh) - b(di - gf) + c(dh - eg)$$

Conveniently, $\det(M) = \prod \varepsilon_i \therefore$ if any $\varepsilon = 0,\ \det(M) = 0$

So:

$M\vec{e}_1 = \varepsilon_1\vec{e}_1$

$(M - \varepsilon_1 I)\vec{e}_1 = 0$

So $|M - \varepsilon_1 I| = 0$

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - cb = 0$$

The eigenvalues are the roots of this characteristic equation

$$\begin{vmatrix} 1 \not> & 3 \\ 3 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 4 = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$$

$\varepsilon_1 = 4 \quad \varepsilon_2 = -2$

Find vectors by examination

$$M - \varepsilon_1 I = \begin{pmatrix} 1-4 & 3 \\ 3 & 1-4 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \vec{e}_1 = 0$$

$\vec{e}_1 = \frac{1}{\sqrt{2}}\binom{1}{1}$,

$$\vec{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

VII. Some matrix operations

Inverse

$$M^{-1}M = I$$

Transpose

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Hermitian transpose: VERY IMPORTANT, whenever we transpose a complex vector, we need to use the Hermitian transpose, or else we will not get real lengths for vectors dotted with themselves

$$M^t = M^H = (M^T)^*$$

$$\begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix}$$

Hermitian and symmetric matrix

$$\text{Symmetric} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{Hermitian} \begin{bmatrix} a+bi & c+id \\ c-id & e+if \end{bmatrix}$$

$$M = M^T \qquad\qquad H^t = H$$

Hermitian matrix will always have real eigenvalues. Hermitian and symmetric matrixes have normal eigenvectors (but not necessarily complete). Projection matrixes are symmetric but only have 1 (and 0) as an eigenvalue with the vector of the projection being the eigenvector.

VIII. Commutation

[A,B] = AB-BA, AB-BA only if they share eigenvectors.

IX. Spectral theorem of Symmetric or Hermitian matrixes.

$$M = \sum_i \varepsilon_i P_{\hat{e}_i}$$

Which means that $M\vec{a}$ is equivalent to weighting the eigenvalues of M by the projection of $\vec{a}$ on the corresponding vector. This will be important to quantum…