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1. Electromagnetism
 - a) Maxwell's Equations

$$\begin{aligned}\vec{\nabla} \cdot \vec{D} &= \rho \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{j}\end{aligned}$$

- b) Constitutive Relations

$$\begin{aligned}\vec{D} &= \epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P} \\ \vec{B} &= \mu \vec{H} = \mu_0 \vec{H} + \vec{M}\end{aligned}$$

- c) Tensor Review

A tensor is a multidimensional array that relates a scalar, vector, or another tensor to another scalar, vector, or tensor by a linear operation in a given set of m space dimensions. Tensors have rank n describing the dimensionality of the multidimensional array and m components along each array. Thus in general a tensor has m^n total components. These components are not necessarily independent and will in general be related to each other based on symmetry conditions of a system in question.

e.g.

A scalar is a rank 0 tensor and thus has 1 single component no matter the number of space dimensions m ($m \geq 1$).

A vector is a rank 1 tensor. In 3 dimensions (x,y,z space) a vector has 3 components.

A square matrix is a rank 2 tensor. The above quantities ϵ and μ are in general rank 2 tensors because they relate two rank 1 tensors to each other.

Higher rank tensors include the piezoelectric tensor which is rank 3 that relates the rank 1 electric field to the rank 2 strain tensor the rank 4 compliance tensor that relates the rank 2 stress tensor to the rank 2 strain tensor.

In tensor form, the electric permittivity ϵ and the magnetic permeability μ can be written as follows:

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix}$$

One can show that these tensors are symmetric for most crystalline media meaning $\epsilon_{12} = \epsilon_{21}$, $\epsilon_{13} = \epsilon_{31}$, $\epsilon_{23} = \epsilon_{32}$, $\mu_{12} = \mu_{21}$, $\mu_{13} = \mu_{31}$, and $\mu_{23} = \mu_{32}$. By finding the eigenvalues and eigenvectors of these tensors, one can find principle axes of the material along which the electric field and displacement field and the magnetic field and induction fields are always parallel.

In addition further symmetry considerations of a particular crystal can reduce the number of independent components. If the crystal is cubic or isotropic, it turns out there is only 1 independent component and the constants are effectively scalars. For this class, we will only consider this particular case where the constants take this simple scalar form.

d) Polarization – Damped Oscillator Model

$$\frac{d^2 \vec{P}}{dt^2} + \sigma \frac{d\vec{P}}{dt} + \omega_0^2 \vec{P} = \omega_0^2 \chi_0 \epsilon_0 \vec{E}$$

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \approx \vec{E}_0 e^{-i\omega t}$$

$$\vec{P} = \frac{\omega_0^2 \epsilon_0 \chi_0}{\omega_0^2 - \omega^2 - i\sigma\omega} \vec{E} = \epsilon_0 \chi(\omega) \vec{E}$$

$$\chi = \chi' + i\chi''$$

$$\epsilon = \epsilon_0(1 + \chi)$$

2. Wave Optics

a) Wave Solution to Maxwell's Equations

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{H} - \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

$$\frac{1}{c^2} = \mu\epsilon$$

$$c = \frac{c_0}{n}$$

$$\mu = \mu_r \mu_0$$

$$\epsilon = \epsilon_r \epsilon_0$$

$$\omega = c|\vec{k}| = \frac{c_0}{n} \frac{2\pi}{\lambda}$$

$$\vec{v}_{phase} = \frac{\omega \vec{k}}{|\vec{k}|^2} = \frac{\omega}{|\vec{k}|} \hat{k} = c \hat{k}$$

$$|v_{group}| = \frac{\partial \omega}{\partial k} = \frac{\partial \omega}{\partial \lambda} \frac{\partial \lambda}{\partial k} = \frac{\partial}{\partial \lambda} \left(\frac{c_0}{n} \frac{2\pi}{\lambda} \right) \frac{\partial \lambda}{\partial k} = \frac{2\pi}{\lambda} c_0 \frac{\partial}{\partial \lambda} \left(\frac{1}{n} \right) - \frac{2\pi c_0}{\lambda^2} \frac{1}{n} = \frac{\lambda c_0}{n^2} \frac{\partial n}{\partial \lambda} + \frac{c_0}{n}$$

The group velocity will have the same magnitude as the phase velocity if the index of refraction does not depend on the wavelength.

b) Optical Constants

$$n = \frac{c_0}{c} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r}$$

In most materials, $\mu_r \approx 1$, so $n' \approx \sqrt{\epsilon_r} \approx \sqrt{1 + \chi} \approx \sqrt{1 + \chi' + i\chi''} \equiv n + i\alpha$
 Here n is the traditional refractive index and α is the absorption coefficient.

c) Poynting Vector & EMR Energy Density

In a charge free medium:

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{H} = 0$$

If $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$ $\vec{H} = \vec{H}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

Then the above equations imply:

$$\vec{k} \cdot \vec{E} = 0$$

$$\vec{k} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} + \mu \frac{\partial \vec{H}}{\partial t} = 0$$

Implies

$$\vec{E} \cdot \vec{H} = 0$$

This means that EMR waves of the given form are transverse waves. This means they propagate in the direction \vec{k} with the two fields oscillating perpendicular to each other and the direction of propagation.

The following conservation equation can be derived

$$\frac{\partial}{\partial t} \int \frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) dV + \int (\vec{E} \times \vec{H}) \cdot \hat{n} dS = 0$$

The volume integral integrand term is the energy that is stored in the electric and magnetic fields per unit volume. The surface integral integrand term is the energy surface flux per area. Their sum must be zero for conservation of energy.

$$\vec{E} \times \vec{H} = \vec{S}$$

\vec{S} is known as the Poynting vector and is in the same direction as the propagating wave.

3. Optical Interfaces

a) Reflection & Snell's Law

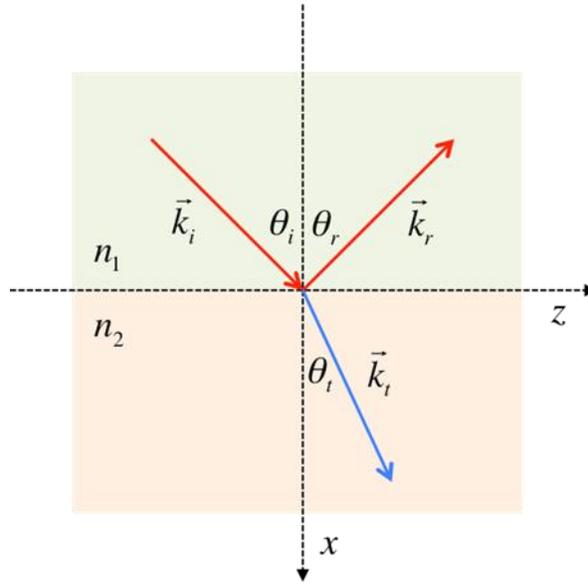
At a given surface interface between two materials or fluids, the boundary conditions between the surface coupled with Maxwell's equations results in the allowed reflection and transmission forms of the EMR that is incident on the surface.

For a charge free interface, the following relations hold between the two materials:

$$\theta_i = \theta_r$$

$$n_1 \sin \theta_i = n_2 \sin \theta_t$$

The second equation is known as Snell's Law, originally experimentally observed, but can be derived a multitude of ways including the above way.



b) Wave Guides

If $n_1 > n_2$, there exists a critical angle at which the transmission angle is 90° . This allows for the construction of a wave guide, where an incoming wave of light can be totally contained in the higher index medium surrounded by the lower index medium. This is known as total internal reflection.

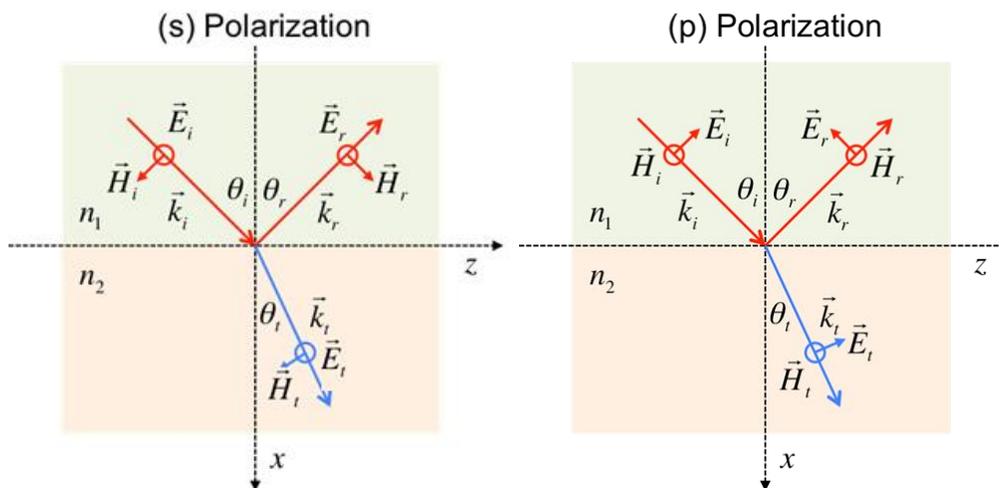
$$\sin \theta_c = \frac{n_2}{n_1}$$

If the wave sent into the wave guide comes from a material of refractive index $n < n_1$, then the maximum angle at which the incident wave can go into the wave guide and not be transmitted out of the waveguide is given by:

$$n \sin \theta_{MAX} = \sqrt{n_1^2 - n_2^2}$$

c) s- and p- Polarized Light

Since EMR waves are transverse waves, the field direction of the wave can be polarized at different angles relative to the interfaces with which the waves interact. By choosing two specific polarizations, one with the electric field parallel to the plane of incidence ((s)-Polarization) and one with the electric field perpendicular to the plane of incidence ((p)-Polarization), any general polarized light interactions can be viewed as a linear combination of these two specific polarizations. By considering the boundary conditions imposed on the fields (electric and magnetic) at the interface for both polarizations, one can derive reflection and transmission coefficients for the EMR waves interacting with the surface.



$$r_s = \left(\frac{E_{ry}}{E_{iy}} \right) = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t}$$

$$r_p = \left(\frac{E_r}{E_i} \right) = \frac{n_2 \cos \theta_i - n_1 \cos \theta_t}{n_2 \cos \theta_i + n_1 \cos \theta_t}$$

$$t_s = \left(\frac{E_{ty}}{E_{iy}} \right) = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t}$$

$$t_p = \left(\frac{E_t}{E_i} \right) = \frac{2n_1 \cos \theta_i}{n_2 \cos \theta_i + n_1 \cos \theta_t}$$

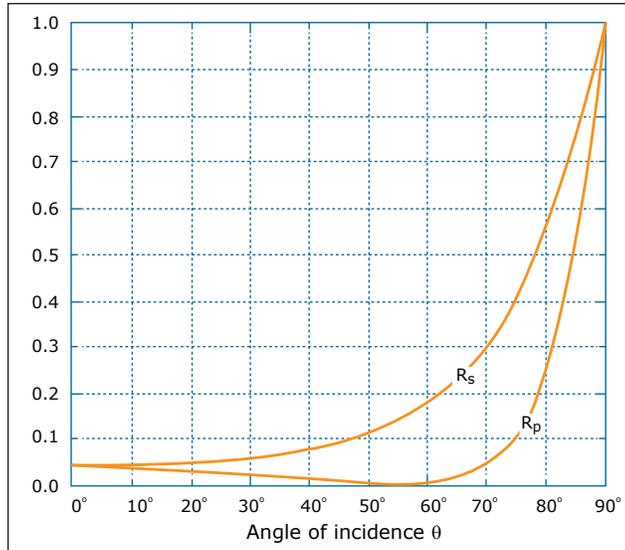


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d) Anti-Reflective Coatings

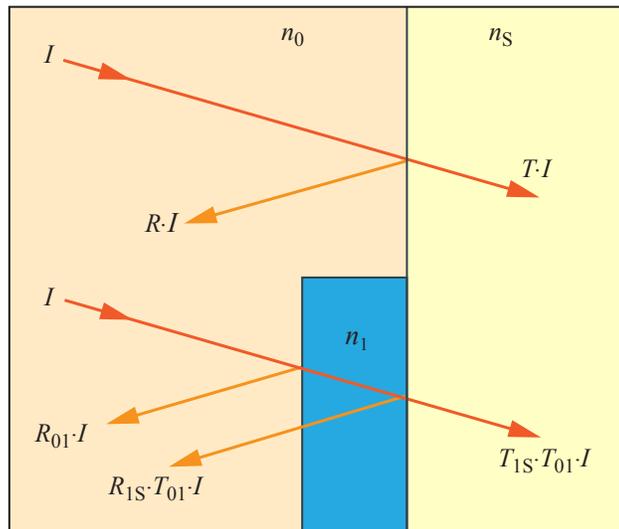


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Minimal Reflection Occurs When Refractive Indices Satisfy the Following:

$$n_I = \sqrt{n_0 n_S}$$

Alternatively, for a specific wavelength, all light can be transmitted using a quarter wave stack:

$$d = \frac{\lambda_0}{4n_I}$$

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3.024 Electronic, Optical and Magnetic Properties of Materials
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