Lecture 23

Layered Materials and Photonic Band Diagrams

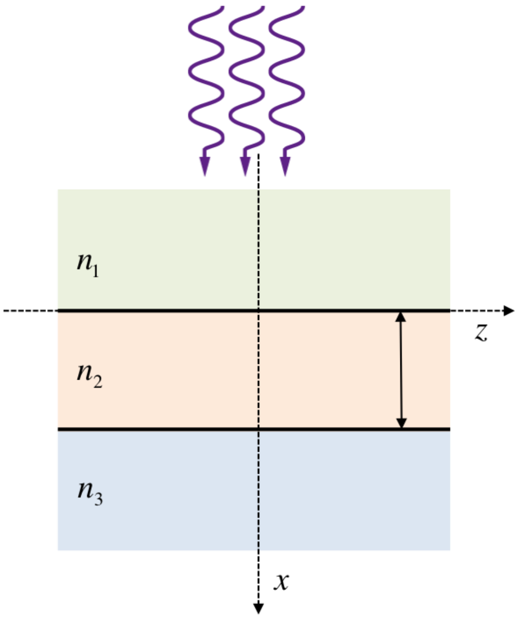
Today

1. Transfer matrix approaches to layered media
2. Periodic layered media
3. Bloch solutions and photonic band diagrams

Questions you should be able to answer by the end of today's lecture

1. How to derive the transfer matrix that describes the field transformation at interfaces?
2. Transfer matrix approach to solving periodic systems of dielectric materials.
3. Eigenvalue problem and its solutions
4. Photonic band diagrams and band gaps

Transfer Matrix approach: general treatment of multilayered optical materials.



Consider a 3-layered material with an index of refraction profile:

$$n(x)=\begin{cases} n_1, & x<0 \\ n_2, & 0<x<d \\ n_3, & d<x \end{cases}$$

As discussed in the previous lecture, the z component of the wavevector $k_z \equiv \beta$ does not change throughout the problem (consequence of phase continuity). This leads to a simple dependence for the evolution in z direction:

$$\vec{E}=\vec{E}(x)e^{i(\omega t-\beta z)}$$

However for x and y directions the scenario is more complicated as we have noticed in the previous lecture when we derived the reflection and transmission coefficients for the single interface. Now we would like to find a general way to relate any pair of electric and magnetic field amplitudes in any layer to those in any other layer.

As usual we would break our electric (and consequently magnetic) field components into ***s***- and ***p***-polarizations. Here we will only consider ***s***-polarized electric field for simplicity (y-direction for *E* and x-z plane, perpendicular to wavevector for *H*).

For ***s***-polarized field:

$$E(x)=\begin{cases} E_1 e^{ik_{1x}x}+E'_1 e^{-ik_{1x}x}, & x<0 \\ E_2 e^{ik_{2x}x}+E'_2 e^{-ik_{2x}x}, & 0<x<d \\ E_3 e^{ik_{3x}(x-d)}+E'_3 e^{-ik_{3x}(x-d)}, & x>d \end{cases}$$

Where $E_1,\ E_2,\ E_3$ are the amplitudes of components of the electric field propagating forward and $E'_1,\ E'_2,\ E'_3$ are the amplitudes of the reflected components propagating backward.

If we were to relate the amplitudes in the layer 1 to the layer 3, we will have to consider propagation across two interfaces 1-2 and 2-3 and the simple propagation through the homogeneous layer 2.

The effect of propagation through a medium of index n_2 and thickness *d* is captured by the propagation matrix:

$$P_2=\begin{pmatrix} e^{ik_{2x}d} & 0 \\ 0 & e^{-ik_{2x}d} \end{pmatrix}$$

Which is just simple phase accumulation for a plane wave propagating distance *d* in a homogeneous medium of index n_2.

The interface matrices can be derived from the boundary conditions akin to the way we treated a single interface:

$$\vec{E}_{1\|} = \vec{E}_{1\|} \Rightarrow E_{1y} + E'_{1y} = E_{2y} + E'_{2y}$$

$$H_{1\|} = H_{2\|} \Rightarrow H_{1z} + H'_{1z} = H_{2z} + H'_{2z} \Rightarrow n_1 \cos\theta_1 E_{1y} - n_1 \cos\theta_1 E'_{1y} = n_2 \cos\theta_2 E_{2y} - n_2 \cos\theta_2 E'_{2y}$$

Then can rewrite the equations above in a matrix form:

$$\left.\begin{array}{c} E_{1y} + E'_{1y} = E_{2y} + E'_{2y} \\ n_1 \cos\theta_1 E_{1y} - n_1 \cos\theta_1 E'_{1y} = n_2 \cos\theta_2 E_{2y} - n_2 \cos\theta_2 E'_{2y} \end{array}\right\} \Rightarrow$$

$$\begin{pmatrix} 1 & 1 \\ n_1\cos\theta_1 & -n_1\cos\theta_1 \end{pmatrix}\begin{pmatrix} E_{1y} \\ E'_{1y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ n_2\cos\theta_2 & -n_2\cos\theta_2 \end{pmatrix}\begin{pmatrix} E_{2y} \\ E'_{2y} \end{pmatrix} \Rightarrow D_1\begin{pmatrix} E_{1y} \\ E'_{1y} \end{pmatrix} = D_2\begin{pmatrix} E_{2y} \\ E'_{2y} \end{pmatrix}$$

$$D_2^{-1}D_1\begin{pmatrix} E_{1y} \\ E'_{1y} \end{pmatrix} = \begin{pmatrix} E_{2y} \\ E'_{2y} \end{pmatrix} \Rightarrow D_{12}\begin{pmatrix} E_{1y} \\ E'_{1y} \end{pmatrix} = \begin{pmatrix} E_{2y} \\ E'_{2y} \end{pmatrix};\ D_{12} = D_2^{-1}D_1$$

Then the matrix D_{12} can be simply found as:

$$D_{12} = D_2^{-1}D_1 = \begin{pmatrix} 1 & 1 \\ n_2\cos\theta_2 & -n_2\cos\theta_2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ n_1\cos\theta_1 & -n_1\cos\theta_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2n_2\cos\theta_2} \\ \frac{1}{2} & -\frac{1}{2n_2\cos\theta_2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ n_1\cos\theta_1 & -n_1\cos\theta_1 \end{pmatrix}$$

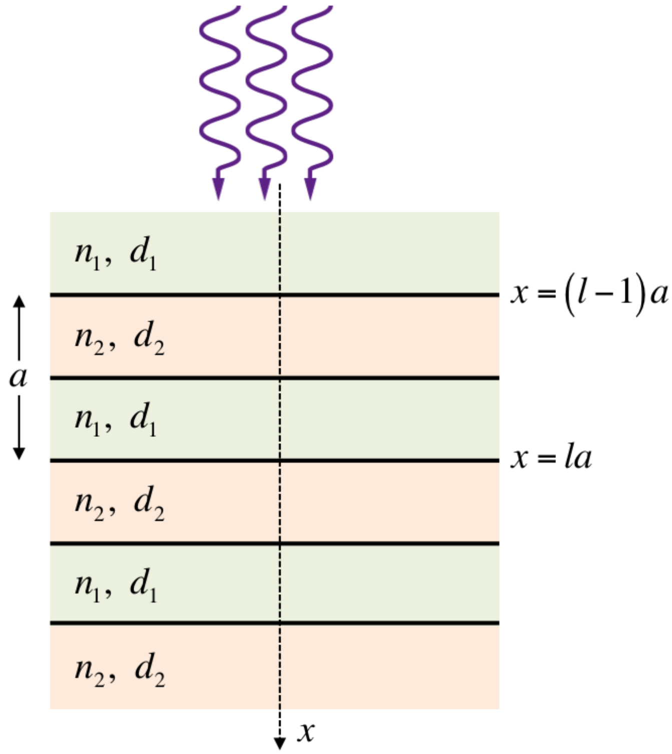
$$D_{12} = \begin{pmatrix} \frac{n_2\cos\theta_2 + n_1\cos\theta_1}{2n_2\cos\theta_2} & \frac{n_2\cos\theta_2 - n_1\cos\theta_1}{2n_2\cos\theta_2} \\ \frac{n_2\cos\theta_2 - n_1\cos\theta_1}{2n_2\cos\theta_2} & \frac{n_2\cos\theta_2 + n_1\cos\theta_1}{2n_2\cos\theta_2} \end{pmatrix}$$

Then in order to get find the connection between the electric field amplitudes in layer 1 and layer 3 we need to do the following:

$$D_{23}P_2D_{12}\begin{pmatrix} E_1 \\ E'_1 \end{pmatrix} = \begin{pmatrix} E_3 \\ E'_3 \end{pmatrix}$$

Periodic Medium

We have previously considered electronic properties of periodic materials. We have found that periodicity led to the gaps in the electronic energy levels. Let's now consider a material that is periodic on the scale similar to the wavelength of light. These materials are called "photonic crystals".



The index of refraction is a periodic function:

$$n(x)=\begin{cases} n_2 & (l-1)a<x<(l-1)a+d_2 \\ n_1 & la-d_1<x<la \end{cases}$$

$$n(x+la)=n(x)$$

$$d_1+d_2=a$$

The solution in each medium has the form:

$$E=E(x)e^{i(\omega t-\beta z)}$$

Focus initially on the x dependence of the solutions:

$$E(x)=\begin{cases} E_{1l}e^{ik_{1x}(x-la)}+E'_{1l}\,e^{-ik_{1x}(x-la)} & la-d_1<x<la \\ E_{2l}e^{ik_{2x}(x-la+d_1)}+E'_{2l}\,e^{-ik_{2x}(x-la+d_1)} & (l-1)a<x<(l-1)a+d_2 \end{cases}$$

Recall that interface matrices D have the following form:

$$D_{ik} = D_k^{-1} D_i = \begin{pmatrix} 1 & 1 \\ n_k \cos\theta_k & -n_k \cos\theta_k \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ n_i \cos\theta_i & -n_i \cos\theta_i \end{pmatrix} = \begin{pmatrix} \dfrac{n_k \cos\theta_k + n_i \cos\theta_i}{2n_k \cos\theta_k} & \dfrac{n_k \cos\theta_k - n_i \cos\theta_i}{2n_k \cos\theta_k} \\ \dfrac{n_k \cos\theta_k - n_i \cos\theta_i}{2n_k \cos\theta_k} & \dfrac{n_k \cos\theta_k + n_i \cos\theta_i}{2n_k \cos\theta_k} \end{pmatrix}$$

To get the matrix multiplication right it is necessary to carefully consider the form of the solution:

$x = la^{(1)}$ denotes the solution in the medium 1 at the interface $x = la$ (Left Hand Side = LHS)

$x = la - d_1^{(1)}$ denotes the solution in the medium 1 at the interface $x = la - d_1$ (RHS)

Consequently:

$$\left.\begin{aligned} E(x)\big|_{x=la^{(1)}} &= E_{1l} + E'_{1l} \\ E(x)\big|_{x=la-d_1^{(1)}} &= E_{1l} e^{-ik_{1x}d_1} + E'_{1l} e^{ik_{1x}d_1} \end{aligned}\right\} \Rightarrow P_1 = \begin{bmatrix} e^{-ik_{1x}d_1} & 0 \\ 0 & e^{ik_{1x}d_1} \end{bmatrix}$$

From the previous lecture we know that relating the fields across the interface:

$$E(x)\big|_{x=la-d_1^{(2)}} = E_{2l} + E'_{2l}$$

$$D_2 \begin{pmatrix} E_{2l} \\ E'_{2l} \end{pmatrix} = D_1 P_1 \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix} \Rightarrow \begin{pmatrix} E_{2l} \\ E'_{2l} \end{pmatrix} = D_2^{-1} D_1 P_1 \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix}$$

Now we are at the LHS of the $x = na - d_1$ interface (i.e. in medium 2), let's propagate the wave in medium 2 to the RHS of interface: $x = la - d_1 - d_2 = (l-1)a$.

$$\left.\begin{aligned} E(x)\big|_{x=la-d_1^{(2)}} &= E_{2l} + E'_{2l} \\ E(x)\big|_{x=(l-1)a^{(2)}} &= E_{2l} e^{-ik_{2x}d_2} + E'_{2l} e^{ik_{2x}d_2} \end{aligned}\right\} \Rightarrow P_2 = \begin{bmatrix} e^{-ik_{2x}d_2} & 0 \\ 0 & e^{ik_{2x}d_2} \end{bmatrix}$$

And once again we match the fields across the interface to extract $E_{1(l-1)}$ and $E'_{1(l-1)}$:

$$E(x)\big|_{x=(l-1)a^{(2)}} = E_{2l} e^{-ik_{2x}d_2} + E'_{2l} e^{ik_{2x}d_2}$$

$$E(x)\big|_{x=(l-1)a^{(1)}} = E_{1(l-1)} + E'_{1(l-1)}$$

$$D_1 \begin{pmatrix} E_{1(l-1)} \\ E'_{1(l-1)} \end{pmatrix} = D_2 P_2 \begin{pmatrix} E_{2l} \\ E'_{2l} \end{pmatrix} \Rightarrow \begin{pmatrix} E_{1(l-1)} \\ E'_{1(l-1)} \end{pmatrix} = D_1^{-1} D_2 P_2 \begin{pmatrix} E_{2l} \\ E'_{2l} \end{pmatrix}$$

Then putting it all together we finally get:

$$\begin{pmatrix} E_{1(l-1)} \\ E'_{1(l-1)} \end{pmatrix} = D_1^{-1} D_2 P_2 D_2^{-1} D_1 P_1 \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix}$$

Now recall the **Bloch theorem** that we have used to get the electronic wavefunction for the periodic material. We can use the same logic for the optically periodic structure or a "photonic crystal".

According to Bloch theorem the solution to a periodically constrained system has to have a form:

$$\begin{pmatrix} E_{1(l-1)} \\ E'_{1(l-1)} \end{pmatrix} = e^{-iKa} \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix}$$

Combining these two equations leads us to the following expression:

$$D_1^{-1} D_2 P_2 D_2^{-1} D_1 P_1 \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix} = e^{-iKa} \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix} \Rightarrow M \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix} = e^{-iKa} \begin{pmatrix} E_{1l} \\ E'_{1l} \end{pmatrix}$$

$$M = D_1^{-1} D_2 P_2 D_2^{-1} D_1 P_1$$

The matrix elements are given by the following expression:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$M_{11} = e^{-ik_{1x}d_1} \left[\cos k_{2x} d_2 - \frac{1}{2} i \left(\frac{k_{1x}}{k_{2x}} + \frac{k_{2x}}{k_{1x}} \right) \sin k_{2x} d_2 \right]$$

$$M_{12} = e^{ik_{1x}d_1} \frac{1}{2} i \left(\frac{k_{1x}}{k_{2x}} - \frac{k_{2x}}{k_{1x}} \right) \sin k_{2x} d_2$$

$$M_{21} = -e^{-ik_{1x}d_1} \frac{1}{2} i \left(\frac{k_{1x}}{k_{2x}} - \frac{k_{2x}}{k_{1x}} \right) \sin k_{2x} d_2$$

$$M_{22} = e^{ik_{1x}d_1} \left[\cos k_{2x} d_2 + \frac{1}{2} i \left(\frac{k_{1x}}{k_{2x}} + \frac{k_{2x}}{k_{1x}} \right) \sin k_{2x} d_2 \right]$$

$$k_{1x} = |k_1| \cos\theta_1 = \frac{\omega n_1 \cos\theta_1}{c_0},\ k_{2x} = |k_2| \cos\theta_2 = \frac{\omega n_2 \cos\theta_2}{c_0},\ |k_1| \sin\theta_1 = |k_2| \sin\theta_2 \equiv \beta$$

$$k_{1x} = \frac{\omega n_1}{c_0} \sqrt{1 - \sin^2\theta_1} = \sqrt{\left(\frac{\omega n_1}{c_0} \right)^2 - \beta^2},\ k_{2x} = \sqrt{\left(\frac{\omega n_2}{c_0} \right)^2 - \beta^2}$$

Each one of the matrix elements depends on ω and β.

Photonic band structures.

The matrix equation above is simply an eigenvalue problem.

For a 2x2 matrix the eigenvalues are given by:

$$e^{-iKa} = \frac{1}{2}\left(M_{11}+M_{22}\right) \pm \sqrt{\frac{1}{4}\left(M_{11}+M_{22}\right)^2 + M_{12}M_{21}}$$

This equation defines the dispersion relations for the Bloch wavenumber K and ω and β.

$$\frac{e^{iKa}+e^{-iKa}}{2} = \cos Ka = \frac{1}{2}\left(M_{11}+M_{22}\right) \Rightarrow K\left(\beta,\omega\right) = \frac{1}{a}\cos^{-1}\left(\frac{M_{11}+M_{22}}{2}\right)$$

Here we have taken into account that matrix M has a determinant of unity (easy to check yourself), which means both e^{iKa} and e^{-iKa} are the eigenvalues.

The eigenvectors then are:

$$\begin{pmatrix} E_{10} \\ E'_{10} \end{pmatrix} = \begin{pmatrix} M_{12} \\ e^{\pm iKa} - M_{11} \end{pmatrix}$$

Now we can distinguish between two qualitative regimes:

$\left|\frac{M_{11}+M_{22}}{2}\right| < 1 \Rightarrow$ K is real, waves are propagating

$\left|\frac{M_{11}+M_{22}}{2}\right| = 1 \Rightarrow$ defines the band edges $\Rightarrow$ cosKa=1 $\Rightarrow K = \frac{m\pi}{a}$

$\left|\frac{M_{11}+M_{22}}{2}\right| > 1 \Rightarrow$ K is complex, waves are evanescent (decaying in amplitude)

Then the general solution:

$$E_{1lK}\left(x\right)e^{-iKla} = \left(E_{10}e^{ik_{1x}(x-la)} + E'_{10}\,e^{-ik_{1x}(x-la)}\right)e^{-iKla}$$

Recall that:

$$k_{1x} = \frac{\omega n_1}{c_0}\sqrt{1-\sin^2\theta_1} = \sqrt{\left(\frac{\omega n_1}{c_0}\right)^2 - \beta^2},\ k_{2x} = \sqrt{\left(\frac{\omega n_2}{c_0}\right)^2 - \beta^2}$$

Lets first consider the case of $\beta = 0$ which corresponds to normal incidence (incidence angle of 0). Then:

$$k_{1x} = \left|k_1\right| = \frac{\omega n_1}{c_0},\ k_{2x} = \left|k_2\right| = \frac{\omega n_2}{c_0}$$

Then the dispersion relation ω vs. K can be expressed as:

$$\cos Ka = \frac{M_{11}+M_{22}}{2} = \frac{1}{2}e^{-ik_1d_1}\left[\cos k_2d_2 - \frac{1}{2}i\left(\frac{k_2}{k_1}+\frac{k_1}{k_2}\right)\sin k_2\boldsymbol{d}_2\right] + \frac{1}{2}e^{ik_1d_1}\left[\cos k_2d_2 + \frac{1}{2}i\left(\frac{k_2}{k_1}+\frac{k_1}{k_2}\right)\sin k_2d_2\right]$$

$$\cos Ka = \cos k_1d_1\cos k_2d_2 - \frac{1}{2}\left(\frac{n_2}{n_1}+\frac{n_1}{n_2}\right)\sin k_1d_1\sin k_2d_2$$

Finally we find:

$$\cos Ka = \cos\frac{\omega n_1d_1}{c_0}\cos\frac{\omega n_2d_2}{c_0} - \frac{1}{2}\left(\frac{n_2}{n_1}+\frac{n_1}{n_2}\right)\sin\frac{\omega n_1\boldsymbol{d}_1}{c_0}\sin\frac{\omega n_2d_2}{c_0}$$

Figure removed due to copyright restrictions. Bloch waves corresponding to the A and B solutions for frequency at the edge of the Brillouin zone: Unknown source.

On the right picture A and B are the Bloch waves corresponding to the A and B solutions for frequency at the edge of the Brillouin zone $\left(K=\frac{g}{2}\right)$. Here $g=\frac{2\pi}{a}$.

When $\beta \neq 0$, i.e. the wave hits the periodic medium at an angle other than zero, we can find plot the band diagram for $\omega(\beta)$:

Figure removed due to copyright restrictions. TM and TE polarization: Unknown source.

Reflection from a dielectric mirror.

Consider a situation where *N* periods are assembled into a stack this arrangement is also called a dielectric or distributed Bragg reflector (DBR) and is used extensively in applications ranging from laser cavities to telecommunication filters.

$$\begin{pmatrix} E_0 \\ E'_0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}^N \begin{pmatrix} E_N \\ E'_N \end{pmatrix}$$

$$R = \left|r_N\right|^2 = \frac{\left|M_{21}\right|^2}{\left|M_{21}\right|^2 + \left(\dfrac{\sin Ka}{\sin NKa}\right)^2}$$

This means we can achieve nearly perfect reflection from a material with a large number of layers:

$$R = \left|r_N\right|^2 = \frac{\left|M_{21}\right|^2}{\left|M_{21}\right|^2 + \left(\dfrac{\sin Ka}{\sin NKa}\right)^2} \xrightarrow{Ka \to 0} \frac{\left|M_{21}\right|^2}{\left|M_{21}\right|^2 + \left(\dfrac{Ka}{NKa}\right)^2} = \frac{\left|M_{21}\right|^2}{\left|M_{21}\right|^2 + \left(\dfrac{1}{N}\right)^2} \xrightarrow{N \to \infty} \frac{\left|M_{21}\right|^2}{\left|M_{21}\right|^2} = 1$$