Dec. 07 2005: Lecture 27:

Eigenfunction Basis

Reading:
Kreyszig Sections: §4.7 (pp:233–38) , §4.8 (pp:240–248)

Sturm-Liouville Theory, Orthogonal Eigenfunctions

The trigonometric functions have the property that they are orthogonal, that is:

$$\begin{aligned}
\int_{x_0}^{x_0+\lambda} \sin\left(\frac{2\pi M}{\lambda}x\right)\sin\left(\frac{2\pi N}{\lambda}x\right)dx &= \begin{cases} \frac{\lambda}{2} \text{ if } M = N \\ 0 \text{ if } M \neq N \end{cases} \\
\int_{x_0}^{x_0+\lambda} \cos\left(\frac{2\pi M}{\lambda}x\right)\cos\left(\frac{2\pi N}{\lambda}x\right)dx &= \begin{cases} \frac{\lambda}{2} \text{ if } M = N \\ 0 \text{ if } M \neq N \end{cases} \\
\int_{x_0}^{x_0+\lambda} \cos\left(\frac{2\pi M}{\lambda}x\right)\sin\left(\frac{2\pi N}{\lambda}x\right)dx &= 0 \text{ for any integers } M, N
\end{aligned} \tag{27-1}$$

This property allowed the Fourier series to be obtained by multiplying a function by one of the basis functions and then integrating over the domain.

extra notes: Inner Products for Functions ....

The orthogonality relation for the trigonometric functions requires two things:

Range The range over which the functions are defined (i.e., values of x for which $f(x)$ and $g(x)$ have a inner product defined) and integrated in their inner product definition.

Inner product The projection operation of one function onto another.

For the trignometric functions, the inner product was a fairly obvious choice:

$$f(x) \cdot g(x) = \int_0^{2\pi} f(x)g(x)dx \tag{27-2}$$

This inner product follows from the *l2-norm* for functions:

$$|f(x)| = \sqrt{\int f(x)f(x)dx} = \sqrt{\int f^2 dx} \tag{27-3}$$

which is one of the obvious ways to measure "the distance of a function from zero." The l2-norm is employed in least-squares-fits.

However, there are different choices of inner products. For example, the Laguerre polynomials (or Laguerre functions) $L_n(x)$ defined by

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right) \tag{27-4}$$

for $0 < x < \infty$ have the orthogonality relation for a weighted inner product:

$$L_n(x) \cdot L_m(x) = \int_0^{\infty} e^{-x} L_n(x)L_m(x)dx = \delta_{mn} \tag{27-5}$$

There are many other kinds of functional norms.

Many ordinary differential equations—including the harmonic oscillator, Bessel, and Legendre—can be written in a general form:

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda p(x)\right] y(x) = 0 \tag{27-6}$$

which is called the *Sturm-Liouville problem.* Solutions to this equation are called eigensolutions for an eigenvalue λ. The function $p(x)$ appears in the orthonormality relation:

$$\int p(x) y_{\lambda_1}(x) y_{\lambda_2}(x) dx = 0 \qquad \text{if } \lambda_1 \neq \lambda_2 \tag{27-7}$$

The same "trick" of multiplying a function by one of the eigensolutions and then summing a series can be used to generate series solutions as a superposition of eigensolutions.

MATHEMATICA® Example: Lecture-27

Legendre functions

Expanding a function as a series of Legendre functions