Nov. 30 2005: Lecture 24:

Systems of Ordinary Differential Equations

Reading:
Kreyszig Sections: §3.1 (pp:152–157) , §3.2 (pp:159–61)

Systems of Ordinary Differential Equations

The ordinary differential equations that have been treated thus far are relations between a single function and how it changes:

$$F(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots, \frac{dy}{dx}, y, x) = 0 \tag{24-1}$$

Many physical models of systems result in differential relations between *several* functions. For example, a first-order system of ordinary differential equations for the functions $(y_1(x), y_2(x), \ldots, y_n(x))$ is:

$$\begin{aligned}
\frac{dy_1}{dx} &= f_1(y_1(x), y_2(x), \ldots, y_n(x), x) \\
\frac{dy_2}{dx} &= f_2(y_1(x), y_2(x), \ldots, y_n(x), x) \\
\vdots &= \vdots \\
\frac{dy_n}{dx} &= f_n(y_1(x), y_2(x), \ldots, y_n(x), x)
\end{aligned} \tag{24-2}$$

or with a vector notation,

$$\frac{d\vec{y}(x)}{dx} = \vec{f}(\vec{y}, x) \tag{24-3}$$

The predator-prey model serves as the classical example of a system of differential equations.

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An iterative example of a predator-prey problem with a slight twist.

Suppose there is a fairly bad joke that circulates around the student population. Students either know the joke or they don't. Of course, freshman enter a constant rate α/year and would have no idea about the joke. If two people meet there are three cases:

Both Clueless Neither of the students know the joke and so the joke does not spread.

Both Jaded Both know the joke and if one begins to tell it, the other interrupts with, "Yeah, Yeah. I heard that one. It's pretty, like, stupid." The joke doesn't spread.

Knows it/Never heard it The student who knows remembers that he or she has a stupid joke to tell, but only thinks to spread the joke with probability ρ/(random student meeting year).

Students have a lot of things on their mind (some of which is education) and so they tend to be forgetful. Students who know the joke tend to forget at rate ϕ/year.

It is closely held secret that Susan Hockfield, MIT's president, has an odd sense of humor. At each commencement ceremony, as the proud candidates for graduation approach the president to collect their hard-earned diploma, President Hockfield whispers to the student, "Have you the joke about...?" If the student says, "Yes. I have heard that joke. It is *very* funny!!!" then the diploma is awarded. However, if the student says, "No. But, I am dying to hear it!!!", the president's face drops into a sad frown and the student is asked to leave without collecting the diploma.[15]

Therefore, an iterative model for the student population that knows the joke is:

$$\text{Naive Fraction(Tomorrow)} = \text{Naive Fraction(Today)} + \text{Change in Naive Fraction}$$

$$\text{Jaded Fraction(Tomorrow)} = \text{Jaded Fraction(Today)} + \text{Change in Jaded Fraction}$$

$$N_{i+1} = N_i + \frac{1}{365}\alpha - \frac{1}{365}\rho N_i J_i$$

$$J_{i+1} = J_i + -\frac{1}{365}\alpha J_i + \frac{1}{365}\rho N_i J_i - \phi J_i$$

1. Write functions that increment the populations each day as functions of α, ρ, and ϕ.
2. Use NestList to create lists of population pairs.
3. Visualize the population evolution for a variety of initial conditions.

The MATHEMATICA® example could be modeled with the set of differential equations:

$$\begin{aligned}\frac{dN}{dt} =& \alpha - \rho NJ \\ \frac{dJ}{dt} =& -\alpha J + \rho NJ - \phi J\end{aligned} \tag{24-4}$$

A *critical point* is one at which the left-hand side of Equations 24-2, 24-3, or 24-4 vanish—in other words, a critical point is a special value of the vector $\vec{y}(x)$ where the system of equations does not evolve. For the system defined by Eq. 24-4, there is one critical point:

$$N(t) = \frac{\alpha + \phi}{\rho} \qquad \text{and} \qquad J(t) = \frac{\alpha}{\alpha + \phi} \tag{24-5}$$

However, while a system that is sitting *exactly* at a critical point will not evolve, not every critical point is *stable* like the one in the MATHEMATICA® example. There are three broad categories of critical points:[16]

Stable Any slight perturbation of the system away from the critical point results in an evolution back to that critical point. In other words, all points in the neighborhood of a stable critical point have a trajectory that is attracted back to that point.

Unstable Some slight perturbation of the system away from the critical point results in an evolution away from that critical point. In other words, some points in the neighborhood of an unstable critical point have trajectories that are repelled by the point.

Circles Any slight perturbation away from a critical point results in an evolution that always remains near the critical point. In other words, all points in the neighborhood of a circle critical point have trajectories that remain in the neighborhood of the point.

Reduction of Higher Order ODEs to a System of First Order ODEs

Higher-order ordinary differential equations can usually be re-written as a *system* of first-order differential equations. If the higher-order ODE can be solved for its largest derivative:

$$\frac{d^n y}{dt^n} = F(\frac{d^{n-1}y}{dt^{n-1}}, \frac{d^{n-2}y}{dt^{n-2}}, \dots, \frac{dy}{dt}, t) \tag{24-6}$$

then $n-1$ "new" functions can be introduced via

$$\begin{aligned}
y_0(t) &\equiv y(t) \\
y_1(t) &\equiv \frac{dy}{dt} = \frac{dy_0}{dt} \\
y_2(t) &\equiv \frac{d^2y}{dt^2} = \frac{dy_1}{dt} \\
\vdots &\equiv \vdots \\
y_{n-1}(t) &\equiv \frac{d^{n-1}y}{dt^{n-1}} = \frac{dy_{n-2}}{dt} \\
y_n(t) &\equiv \frac{d^n y}{dt^n} = F(\frac{d^{n-1}y}{dt^{n-1}}, \frac{d^{n-2}y}{dt^{n-2}}, \dots, \frac{dy}{dt}, t) = \frac{dy_{n-1}}{dt}
\end{aligned} \tag{24-7}$$

or

$$\frac{d}{dt}\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ F(y_{n-1}, y_{n-2}, \dots, y_1, y_0, t) \end{pmatrix} \tag{24-8}$$

[16]There are others that will be discussed later.

For example, the damped harmonic oscillator, $M\ddot{y} + \eta l_o \dot{y} + K_s y = 0$, can be re-written by introducing the momentum variable, $p = Mv = M\dot{y}$, as the system:

$$\begin{aligned} \frac{dy}{dt} &= \frac{p}{M} \\ \frac{dp}{dt} &= -K_s y - \eta l_o p \end{aligned} \tag{24-9}$$

which has only one critical point $y = p = 0$.

The equation for a free pendulum, $MR^2\ddot{\theta} + MgR\sin(\theta) = 0$, can be re-written by introducing the angular momentum variable, $\omega = M \quad \dot{\theta}$ as the system,

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{\omega}{MR} \\ \frac{d\omega}{dt} &= -Mg\sin(\theta) \end{aligned} \tag{24-10}$$

which has two *different* kinds of critical points: $(\omega = 0, \theta = n_{even}\pi)$ and $(\omega = 0, \theta = n_{odd}\pi)$.

Finally, the beam equation $EI\frac{d^4y}{dx^4} = w(x)$ can be rewritten as the system:

$$\frac{d}{dx}\begin{pmatrix} y \\ m_{slope} \\ M \\ S \end{pmatrix} = \begin{pmatrix} m_{slope} \\ \frac{M}{EI} \\ S \\ w(x) \end{pmatrix} \tag{24-11}$$

where m_{slope} is the slope of the beam, M is the local bending moment in the beam, S is the local shearing force in the beam, and $w(x)$ is the load density.

This beam equation does not have any interesting critical points.

Linearization of Systems of ODEs

The critical point plays a very important role in understanding the behavior of non-linear ODEs.

The general autonomous non-linear ODE can be written as:

$$\frac{d\vec{y}}{dt} \equiv \frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} F_1(y_1, y_2, \ldots, y_n) \\ F_2(y_1, y_2, \ldots, y_n) \\ \vdots \\ F_n(y_1, y_2, \ldots, y_n) \end{pmatrix} \equiv \vec{F}(y_1, y_2, \ldots, y_n) \tag{24-12}$$

The fixed points are the solutions to:

$$\vec{F}(y_1^f, y_2^f, \ldots, y_n^f) = \begin{pmatrix} F_1(y_1^f, y_2^f, \ldots, y_n^f) \\ F_2(y_1^f, y_2^f, \ldots, y_n^f) \\ \vdots \\ F_n(y_1^f, y_2^f, \ldots, y_n^f) \end{pmatrix} = \vec{0} \tag{24-13}$$

If the fixed points can be found, then the behavior *near* the fixed points can be analyzed by linearization. Letting $\vec{\delta} = \vec{y} - \vec{y}^f$ be a point near a fixed point, then a linear approximation is:

$$\frac{d}{dt}\vec{\delta} = \underline{J}\delta \tag{24-14}$$

where

$$\underline{J} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}\big|_{\vec{y}^f} & \frac{\partial F_2}{\partial y_1}\big|_{\vec{y}^f} & \cdots & \frac{\partial F_n}{\partial y_1}\big|_{\vec{y}^f} \\ \frac{\partial F_1}{\partial y_2}\big|_{\vec{y}^f} & \frac{\partial F_2}{\partial y_2}\big|_{\vec{y}^f} & \cdots & \frac{\partial F_n}{\partial y_2}\big|_{\vec{y}^f} \\ \frac{\partial F_1}{\partial y_3}\big|_{\vec{y}^f} & \frac{\partial F_2}{\partial y_3}\big|_{\vec{y}^f} & \ddots & \vdots \\ \frac{\partial F_1}{\partial y_n}\big|_{\vec{y}^f} & \cdots & \cdots & \frac{\partial F_n}{\partial y_n}\big|_{\vec{y}^f} \end{pmatrix} \tag{24-15}$$

Equation 24-14 looks very much like a simple linear first-order ODE. The expression

$$\vec{y}(t) = e^{\underline{J}t}\vec{y}(t=0) \tag{24-16}$$

might solve it if the proper analog to the exponential of a matrix were known.

Rather than solve the matrix equation directly, it makes more sense to transform the system into one that is diagonalized. In other words, instead of solving Eq. 24-14 with Eq. 24-15 near the fixed point, find the eigenvalues, λ_i, of Eq. 24-15 and solve the simpler system by transforming the $\vec{\delta}$ into the eigenframe $\vec{\eta}$:

$$\begin{aligned} \frac{d\eta_1}{dt} &= \lambda_1\eta_1 \\ \frac{d\eta_2}{dt} &= \lambda_2\eta_2 \\ \vdots &= \vdots \\ \frac{d\eta_n}{dt} &= \lambda_n\eta_n \end{aligned} \tag{24-17}$$

for which solutions can be written down immediately:

$$\begin{aligned} \eta_1(t) &= \eta_1(t=0)e^{\lambda_1 t} \\ \eta_2(t) &= \eta_2(t=0)e^{\lambda_2 t} \\ \vdots &= \vdots \\ \eta_n(t) &= \eta_n(t=0)e^{\lambda_n t} \end{aligned} \tag{24-18}$$

If any of the eigenvalues of $\underline{J}$ are positive, then an initial condition near that fixed point will diverge from that point—stability occurs only if all the eigenvalues are negative.

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Analyzing Stability for the Predator-Prey Problem
Analyzing the MIT Joke