Massachusetts Institute of Technology

Mathematical Methods for Materials Scientists and Engineers

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Problem Set 4: Due Fri. Nov. 4, Before 5PM: email to the TA.

Individual Exercise I4-1

Kreyszig MATHEMATICA® *Computer Guide:* problem 6.14, page 78

Individual Exercise I4-2

Kreyszig MATHEMATICA® *Computer Guide:* problem 6.16, page 78

Individual Exercise I4-3

Kreyszig MATHEMATICA® *Computer Guide:* problem 7.12, page 87

Individual Exercise I4-4

Kreyszig MATHEMATICA® *Computer Guide:* problem 8.10, page 96

Individual Exercise I4-5

Kreyszig MATHEMATICA® *Computer Guide:* problem 8.22, page 96

Group Exercise G4-1

The shape of the catenary

$$y(x) = A \cosh\left(\frac{x+B}{A}\right)$$

is very important. The catenary is the shape of a flexible chain at equilibrium and the rotation of the catenary around $y = 0$ creates a surface of revolution called the *catenoid*.

In the absence of gravity, a soap film suspended between two rings with radii R_1 and R_2, axes lying along $y = 0$, and separated by distance L has a catenoid shape.

Consider a soap film suspended between two identical concentric rings of radius R and separated by distance L. Let the soap film have surface tension γ. Surface tension has units energy/area.

1. Find a parametric representation of the catenoid.
2. The mean curvuture of a surface is the sum of two curvatures. These two curvatures are obtained by slicing the surface with two orthogonal planes—creating two curves—and then using the formula for curvature for a curve. One of the curvatures is simply $1/y(x)$; the second can be obtained by using the result in Kreyszig page 443. Calculate the total mean curvature $\kappa(x)$ of the catenary and plot it.
3. Write a function that calculates the constants A and B given R and L. What are the conditions that there is one solution, two solutions, no solutions?
4. Write a function that calculates the total surface energy, $E(R, L)$, of a soap film.

   The equation for the area of a surface of revolution is:

   $$A[y(x)] = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + (\frac{dy}{dx})^2} dx$$

   Plot the normalized energy surface(s) $E(R, L)/(\gamma RL)$.

Group Exercise G4-2

The diffusion equation

$$\frac{\partial c}{\partial t} = D\nabla^2 c$$

describes how the concentration field $c(\vec{r}, t)$ changes with time proportional to spatial second derivatives. A solution to the diffusion equation requires that *initial conditions* and *boundary conditions* be specified. Boundary conditions specify how $c(\vec{r}, t)$ behaves at particular points in space for all times. Initial conditions specify how $c(\vec{r}, t)$ behaves throughout all space at a particular time.

For some boundary conditions (BCs) and initial conditions (ICs), it is possible to write a solution to the diffusion equation in terms of an integral. For solutions in the infinite domain, the following BCs and ICs are a pair of such conditions,

$$c(x = \pm\infty, y = \pm\infty, z = \pm\infty, t) = 0 \tag{1}$$

$$c(x, y, z, t = 0) = \begin{cases} c_0 & \text{if} |x| \leq \frac{a}{2} \text{ and } |y| \leq \frac{b}{2} \text{ and } |z| \leq \frac{c}{2} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where a, b, and c are finite (i.e., the initial conditions have uniform concentration, c_0, inside a rectangular box and zero outside.

1. Show that

$$c(x, y, z, t) = \int_{\frac{-a}{2}}^{\frac{a}{2}} \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{\frac{-c}{2}}^{\frac{c}{2}} \frac{c_0 d\zeta d\eta d\chi}{(4\pi D t)^{3/2}} e^{-\frac{(x-\chi)^2+(y-\eta)^2+(z-\zeta)^2}{4Dt}} \tag{3}$$

   always satisfies the diffusion equation (independent of BCs and ICs).

2. Show that Eq. 3 always satisfies the boundary conditions, independent of the ICs.

3. Find the closed form of $c(x, y, z, t)$ that satisfies both Eq. 1 and 2.

4. Show by a graphical means that $c(x, y, z, t)$ plausibly approaches the ICs (Eq. 2) as $t \to 0$.

5. Show that the total number of atoms is conserved for $c(x, y, z, t)$.

Group Exercise G4-3

The potential energy of two small magnetic dipoles $\vec{\mu_1}$ and $\vec{\mu_2}$ located at points $\vec{r_1}$ and $\vec{r_2}$ are given by

$$U(\vec{r_1}, \vec{r_2}) = \frac{\mu_o}{4\pi} \left\{ \frac{\vec{\mu_1} \cdot \vec{\mu_2}}{|\vec{r_1} - \vec{r_2}|^3} - \frac{3[\vec{\mu_1} \cdot (\vec{r_1} - \vec{r_2})][\vec{\mu_2} \cdot (\vec{r_1} - \vec{r_2})]}{|\vec{r_1} - \vec{r_2}|^5} \right\}$$

Suppose the first magnetic dipole is located at the origin and points towards the z-direction.

1. Illustrate the potential energy of the two-dipole system as a function of the second magnet's position $\vec{r_2}$ if it is also directed towards the z-direction.

2. Illustrate the potential energy of the two-dipole system if the second magnet is fixed at the location $\vec{r_2}$ but is rotated by θ about the normal to the plane containing both magnets and the z-axis.

3. Illustrate the potential energy of the two-dipole system as a function *both* the second magnet's position $\vec{r_2}$ and its rotation θ about the normal to the plane containing both magnets and the z-axis.

4. Suppose the second magnet is moved along a trajectory, $(x, y, z) = r_0(\cos(2\pi t), \sin(2\pi t), 0)$, and the magnet is always directed towards the trajectory's tangent. Calculate and illustrate the potential energy and the rate of work done on the system as a function of time.

5. **Extra Credit:** Suppose the two magnets are immersed in a viscous fluid and the first magnet is fixed as above. The rate of rotation is given by (approximately)

$$\frac{d\theta}{dt} = \frac{\tau}{4\pi\eta R^2 L}$$

   where R and L are the radius and length of the cylindrical magnet and η is the viscosity in the fluid medium. τ is the torque applied to the magnet.

   The velocity is given by (very approximately)

$$\frac{d\vec{r}}{dt} = \frac{\vec{F}}{6\pi\eta R}$$

   where $\vec{F}$ is the force applied to the magnet.

   Graphically illustrate the position of the rod as a function of time, if the rod is initially at rest at $t = 0$ and located at $\vec{r} = r_0$ for the following initial inclination angles:

$$\theta = (0°, 1°, 45°, 89°, 90°, 91°, 135°, 179°, 180°, 181°, 225°, 269°, 270°, 271°, 315°, 359°)$$