Thermodynamics of Materials 3.00
Example Problems for Week 2

Example Problem 2.1

A closed system consisting of an elastic membrane enclosing a colloidal suspension is squeezed. The compressive pressure is $10Pa$ and the volume of the system changes from $100L$ to $80L$. During this process $1J$ of heat is released. Calculate the change in internal energy of the system.

Solution 2.1

This is a closed system so heat can cross the boundary but matter can't. The problem is a straightforward application of the First Law of Thermodynamics. Which states that the internal energy of a system increases if heat is added to the system and decreases when work is done by the system or $dU = dQ - dW$. This mathematical statement implies that heat added to the system is positive and work done by the system is positive. Note: I've used a sign convention that is different from that in Denbigh. The only work is PdV work so $dU = dQ - PdV$.

$$dU = dQ - dW$$
$$dU = -1J - 10Pa(80L - 100L)\frac{1m^3}{10^3L}$$
$$dU = -1J + 0.02J = 0.98J$$

So the internal energy of the system increases by $0.98J$.

Example Problem 2.2

Two springs, A and B, are attached to each other and to opposite rigid walls separated by a distance L. The point of attachment,x_i , is such that the system of springs is out of equilibrium. The two springs have different spring constants, k_A and k_B. Assume that the two springs make an adiabatic system. Calculate the change in internal energy and force on each spring once the system equilibrates.

Solution 2.2

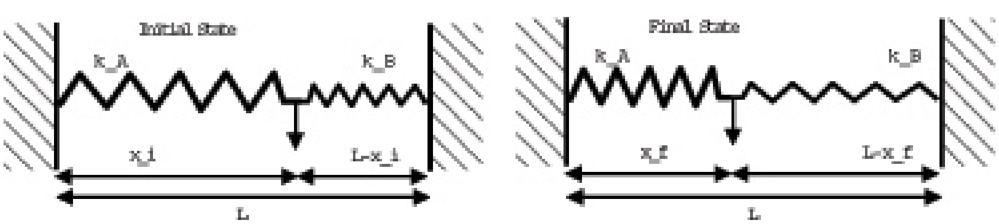

Figure 1: Initial and final states of adiabatic spring system.

The mechanical energy stored in a spring is given by $E = \frac{1}{2}k(x - x_0)^2$ where x is the length of the spring and x_0 is the equilibrium free length of the spring. So the total energy stored in the system of springs is given below where $x_{A,0}$ and $x_{B,0}$ are the equilibrium lengths of each free spring.

$$E = \frac{1}{2}k_A(x_A - x_{A,0})^2 + \frac{1}{2}k_B(x_B - x_{B,0})^2$$

If the length of spring A is x_A and then the length of spring B is $L - x_A$.The equilibrium state of the system is the one with the lowest energy. This is equivalent to calculating $\frac{dE}{dx_A}|_{x_A=x_f}$ and solving for x_f.

$$\begin{aligned}
E &= \frac{1}{2}k_A(x_A - x_{A,0})^2 + \frac{1}{2}k_B(L - x_A - x_{B,0})^2 \\
\frac{dE}{dx_A}|_{x_A=x_f} &= 0 = k_A(x_f - x_{A,0}) - k_B(L - x_f - x_{B,0}) \\
(k_A + k_B)x_f &= k_A x_{A,0} + k_B(L - x_{B,0}) \\
x_f &= \frac{k_A x_{A,0} + k_B(L - x_{B,0})}{(k_A + k_B)}
\end{aligned}$$

Note that there is an upper and lower bound on the equilibrium position,x_f. Mathematically, $0 < x_f < L$. However, each spring has a finite volume and compressibility so the limits on x_f are closer together in a better approximation.

The force on a spring is given by $F = -\frac{dE}{dx} = k(x - x_0)$.

$$\begin{aligned}
F_A &= -k_A(x_f - x_{A,0}) \\
F_A &= -k_A(\frac{k_A x_{A,0} + k_B(L - x_{B,0})}{(k_A + k_B)} - x_{A,0}) \\
F_A &= -\frac{k_A k_B(L - x_{A,0} - x_{B,0})}{k_A + k_B}
\end{aligned}$$

and

$$\begin{aligned}
F_B &= -k_B(L - x_f - x_{B,0}) \\
F_B &= -k_B(L - \frac{k_A x_{A,0} + k_B(L - x_{B,0})}{(k_A + k_B)} - x_{B,0}) \\
F_B &= \frac{k_A k_B(L - x_{A,0} - x_{B,0})}{k_A + k_B}
\end{aligned}$$

Note that the forces on springs A and B are equal in magnitude but opposite in direction. We could have found x_f using this definition of equilibrium. The mathematics that result are exactly the same.

Now substitution of the final and initial positions into the energy expressions will yield the change in internal energy for each spring.

$$\begin{aligned}
\Delta E_A &= E_{A,f} - E_{A,i} \\
\Delta E_A &= \frac{1}{2}k_A(x_f - x_{A,0})^2 + \frac{1}{2}k_A(x_i - x_{A,0})^2 \\
\Delta E_A &= \frac{1}{2}k_A\left[(\frac{k_A x_{A,0} + k_B(L - x_{B,0})}{(k_A + k_B)} - x_{A,0})^2 - (x_i - x_{A,0})^2\right]
\end{aligned}$$

and

$$\Delta E_B = E_{B,f} - E_{B,i}$$

$$\Delta E_B = \frac{1}{2}k_B(L - x_f - x_{B,0})^2 + \frac{1}{2}k_B(L - x_i - x_{B,0})^2$$

$$\Delta E_B = \frac{1}{2}k_B\left[(L - \frac{k_A x_{A,0} + k_B(L - x_{B,0})}{(k_A + k_B)} - x_{B,0})^2 + (L - x_i - x_{B,0})^2\right]$$

Note that in general the change in internal energy of spring A is different from that for spring B. The total change in internal energy of the system of springs is not positive.