

24.118 – Paradox and Infinity
Problem Set 6: Ordinal Arithmetic

You will be graded both on the basis of whether your answers are correct and on the basis of whether they are properly justified. There is no word limit.

Definitions

- \emptyset is the empty set.
- if x and y are sets, then:
 - $x - y$ is the set of all elements of x which are not also elements of y .
 - x is a subset of y , and we write $x \subset y$, just in case any member of x is a member of y .
 - $x \cup y$ is the set of objects that are either elements of x or elements of y (or both).
 $x \cap y$ is the set of objects that are both elements of x and elements of y .
 - $\mathcal{P}(x)$ is the set of all subsets of x
- A set x is *transitive* just in case every member of a member of x is a member of x .
[*Example:* The sets $\{a, b\}$, $\{a, \{a\}\}$ and $\{a, b, \{a\}, \{a, b\}\}$ are all transitive, but the sets $\{a, \{b\}\}$, $\{a, \{\{b\}\}\}$ and $\{\{a\}\}$ are not. (Here I assume that a and b have no elements, and that $a \neq b$.)]
- A *total ordering* is a pair $\langle A, < \rangle$ where A is a set and $<$ is a binary relation on A such that:
 1. For all $x, y, z \in A$, if $x < y$ and $y < z$, then $x < z$ (*transitivity*).
 2. For all $x, y \in A$, if $x \neq y$, then either $x < y$ or $y < x$ (*trichotomy* or *connectivity*).
 3. For all $x \in A$, it is not the case that $x < x$ (*irreflexivity*).

If $\langle A, < \rangle$ is a total ordering we sometimes say that $<$ is a total ordering on A . Note that if $B \subset A$ and $<$ is a total ordering on A , then $<$ is a total ordering on B .

[*Example:* $\langle \mathbb{R}, <_{\mathbb{R}} \rangle$ and $\langle \mathbb{R}, <_{\star} \rangle$ are both total orderings, where \mathbb{R} is the set of real numbers, $<_{\mathbb{R}}$ is the natural ordering of the real numbers (i.e. $x <_{\mathbb{R}} y$ is true just in case $y = x + z$ for some positive number z), and $<_{\star}$ is the inverse of the natural ordering of the natural numbers (i.e. $x <_{\star} y$ is true just in case $x = y + z$ for some positive number z). On the other hand $\langle \mathcal{P}(A), \subset \rangle$ is not a total order, where $\mathcal{P}(x)$ and \subset are defined as above and A is a set with more than one element.]

- A *well-ordering* is a total ordering $\langle A, < \rangle$ such that for any $B \subset A$, if $B \neq \emptyset$, then B has an $<$ -least element, i.e. there is $x \in B$ such that if $y \in B$ and $x \neq y$, then $x < y$.

[*Example:* $\langle \mathbb{N}, <_{\mathbb{N}} \rangle$ is a well-ordering, where \mathbb{N} is the set of natural numbers and $<_{\mathbb{N}}$ is the natural ordering of the natural numbers (i.e. $x <_{\mathbb{N}} y$ is true just in case $y = x + z$ for some positive number z). On the other hand, $\langle \mathbb{R}, <_{\mathbb{R}} \rangle$ is not a well-ordering, where \mathbb{R} and $<_{\mathbb{R}}$ are defined as above.]

- A set is an *ordinal* just in case it is transitive and well-ordered by \in .

[*Example:* The set $\{\emptyset\}$ is an ordinal. $\{\emptyset\}$ is *transitive* because its only member is \emptyset , and \emptyset has no elements; so, trivially every member of every member of $\{\emptyset\}$ (i.e. every member of \emptyset) is a member of $\{\emptyset\}$. \in is a *total ordering* of $\{\emptyset\}$ because each of the following is (trivially) true: (1) if $\emptyset \in \emptyset$ and $\emptyset \in \emptyset$, then $\emptyset \in \emptyset$, (2) if $\emptyset \neq \emptyset$, then either $\emptyset \in \emptyset$ or $\emptyset \in \emptyset$, and (3) it is not the case that $\emptyset \in \emptyset$. \in is a *well-ordering* of $\{\emptyset\}$ because $\{\emptyset\}$ is the only non-empty subset of $\{\emptyset\}$; so, trivially, every non-empty subset of $\{\emptyset\}$ has a \in -least element.]

- if α and β are ordinals, α is smaller than β (in symbols $\alpha < \beta$) just in case $\alpha \in \beta$.¹

Note: In this homework we will be concerned only with pure sets. A set x is *pure* if every member of x is a set, every member of a member of x is a set, every member of a member of a member of x is a set, and so forth. So, for instance, the set of sets of elephants is not a pure set. (Formally, x is pure if the smallest transitive set with x as a subset consists entirely of sets.)

Problems:

1. Ordinals (11 points)

- Say of each of the following whether it is transitive and whether it is well-ordered by \in . (5 Points)
 - \emptyset
 - $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$
 - $\mathcal{P}(\mathcal{P}(\emptyset)) - \{\emptyset\}$
 - $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \{\{\emptyset\}\}$
 - $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \{\{\{\emptyset\}\}\}$
- Show that every member of an ordinal is an ordinal. (2 points)
- Let A be a transitive set of ordinals such that for all $x, y \in A$, if $x \neq y$ then either $x \in y$ or $y \in x$. Show that A is an ordinal. (2 points)

¹It is worth keeping in mind that the notion of smallness used here (which is an *ordinal* notion of smallness, and means something like precedence) is *different* to from use of smallness which we used when we compared sizes of infinity (which is a *cardinal* notion of smallness, and consists in there being a one-one correspondence from the smaller set to a subset of the bigger one but not vice-versa).

(d) If α is an ordinal, we let $\alpha' = \alpha \cup \{\alpha\}$. We say that α' is the *successor* of α . Show that if α is an ordinal, so is α' . (2 points)

2. **Omega** (5 points)

We can use ordinals to define the natural numbers, by letting $0 = \emptyset$, $1 = 0'$, $2 = 1'$ and so on. Let ω be the set of all natural numbers.

(a) Show that \emptyset is the smallest ordinal. (Here and below I mean ‘smallest’ in the ordinal sense. See footnote 1.)

(b) Show that ω is an ordinal.

(c) Show that ω is the smallest infinite ordinal.²

(d) Show that $\omega \neq \omega'$.

(e) Show that ω can be put in one-one correspondence with ω' .

3. **Limit ordinals** (4 points)

If there is β such that $\alpha = \beta'$, we say that α is a *successor* ordinal. An ordinal is a *limit ordinal* just in case it is not a successor ordinal.

Which of the following are limit ordinals?

(a) 0

(b) 17

(c) the set of ordinals smaller than 17

(d) ω

4. **Ordinal arithmetic** (13 points):

We can generalize many of the usual arithmetic operations on natural numbers to all ordinal numbers. For example, we can define addition and multiplication as follows, where α and β are ordinals and λ is a limit ordinal:

- $\alpha + 0 = \alpha$ and $\alpha + \beta' = (\alpha + \beta)'$
- $\alpha + \lambda$ is the set of all $x \in \alpha + \beta$ for $\beta < \lambda$.
- $\alpha \cdot 1 = \alpha$ and $\alpha \cdot \beta' = (\alpha \cdot \beta) + \alpha$
- $\alpha \cdot \lambda$ is the set of all $x \in \alpha \cdot \beta$ for $\beta < \lambda$

(Intuitively, $\alpha + \beta$ is the result of writing an α -sequence followed by a β -sequence, and $\alpha \cdot \beta$ is the result of writing a β -sequence and replacing each element with an α -sequence.)

²An ordinal α is *infinite* just in case $n < \alpha$ for each natural number n .

(a) Give an example of a limit ordinal that is greater than ω . (3 points)

(b) Which of the following are true? (10 points)

- | | |
|---|---|
| i. $\alpha + 1 = \alpha'$ | ii. $1 + \omega = \omega + 1$ |
| iii. $1 + 3 = 3 + 1$ | iv. $(\omega + 2) + \omega < (\omega + \omega) + 3$ |
| v. $1 \cdot 3 = 3 \cdot 1$ | vi. $2 \cdot \omega = \omega + \omega$ |
| vii. $\omega \cdot 2 = \omega + \omega$ | viii. $(\omega \cdot 2) + \omega < (\omega \cdot \omega) + 2$ |
| ix. $\omega \cdot \omega < \omega \cdot (2 \cdot \omega)$ | x. $\omega \cdot (\omega + \omega) = (\omega \cdot \omega) + (\omega \cdot \omega)$ |

Feel free to assume that ordinal addition and multiplication are associative; you may assume, in other words, that $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and that $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$.

5. **Extra Credit** (10 points)

You can describe $\omega + 1$ as a copy of ω followed by 1. Similarly, you can describe $\omega + \omega$ as a copy of ω followed by another copy of ω . In similar terms, describe the following ordinals:

- | | |
|--|--|
| i. $(\omega \cdot 2) + \omega$ | ii. $\omega + (\omega \cdot 2)$ |
| iii. $\omega \cdot \omega$ | iv. $(\omega \cdot \omega) \cdot \omega$ |
| v. $(\omega \cdot \omega) \cdot (\omega \cdot \omega)$ | |

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