

24.118 – Paradox and Infinity  
Problem Set 3: Foundations of Probability

You will be graded both on the basis of whether your answers are correct and on the basis of whether they are properly justified. There is no word limit.

1. Suppose there is an urn with 99 white balls and one red ball. If you choose a ball at random, what is the probability that it will be white? Answer:  $99/100$ , of course! Now suppose you draw again. What is the probability that your next draw will be white? Answer: *it depends on whether you put the first ball back into the urn after the initial draw*. If you did put the first ball back into the urn, the answer is still  $99/100$ . If you didn't, the answer depends on the result of the first draw. If the first draw was white, the answer is  $98/99$ ; if the first draw was red, the answer is  $99/99 = 1$ .

The difference between these two cases is that in the first—when you put the first ball back into the urn—the probability of getting White on the second draw is *independent* of whether you got White on the first draw; in the second case—when you didn't put the first ball back into the urn—the probability of getting White on the second draw depends on whether you got White on the first draw.

In each of the following cases, figure out whether different events are dependent or independent of each other, and use your conclusion to answer the question. (3 points each)

- (a) In the urn example above, suppose you draw 50 balls at random, one after the other? What is the probability that one of the 50 balls you drew was red?(Assume you don't put balls back into the urn after drawing them.)
- (b) You toss a fair coin 5 consecutive times. What is the probability of getting exactly 2 heads.
- (c) Suppose a fair coin is tossed 10 consecutive times? Which of the following outcomes is more likely?
  - T T T T T T T T T T
  - T H H T H T H T H H
- (d) Consider a standard deck of 52 cards. Suppose you are dealt 5 cards. What is the probability of getting at least 2 aces? What is the probability of getting no aces?
- (e) In a standard deck of cards, half the cards are red (R) and half the cards are black (B). Suppose you are dealt 10 cards. Which of the following outcomes is more likely?

- B B B B B B B B B B
- B R R B R B R B R R

2. Let  $A$  be an action you are in a position to perform, and suppose that performing  $A$  might lead to  $k$  different outcomes:  $O_1, O_2, \dots, O_k$ . For each outcome  $O_i$ , we let the probability that outcome  $O_i$  will occur given the you perform  $A$  be  $P(O_i|A)$ . In addition, we let  $v(O_i)$  be a real number corresponding to how much you *value* outcome  $O_i$ . Then the *expected value* of  $A$  is defined as follows

$$EV(A) = \sum_{1 \leq i \leq k} P(O_i|A) \cdot v(O_i)$$

*Example:* Let  $A$  be the action of tossing a coin, and let the two possible outcomes be Heads ( $H$ ) and Tails ( $T$ ). The coin is fair, so  $P(H|A) = 1/2 = P(T|A)$ . You have bet on tails, however, so whereas  $v(T) = 1$ ,  $v(H) = -1$ . The expected value of tossing the coin is 0:

$$EV(A) = P(H|A) \cdot v(H) + P(T|A) \cdot v(T) = 1/2 \cdot 1 + 1/2 \cdot -1 = 0$$

For the purposes of answering the following questions, work on the assumption that the value of an outcome is  $n$  just in case that outcome will result in your receiving  $n$  dollars. (5 points each)

- Consider the following game. You are dealt five cards from a standard 52 card deck. If you get two aces or more, you will receive \$100. Otherwise you will loose \$1. What is the expected value of playing this game?
  - Consider the following game. A coin will be tossed until it lands heads. (If the coin never lands heads, the game is off.) If it lands heads on the first toss, you receive \$1; if it lands heads on the second toss, you receive \$2; and, in general, you receive \$ $n$  if the coin lands heads on the  $n$ th toss). What is the expected value of playing this game?
  - Consider the following game. A coin will be tossed until it lands heads. (If the coin never lands heads, the game is off.) If it lands heads on the first toss, you receive \$2; if it lands heads on the second toss, you receive \$4; and, in general, you receive \$ $2^n$  if the coin lands heads on the  $n$ th toss). What is the expected value of playing this game?
3. Suppose, as before, that the value of an outcome is  $n$  just in case that outcome will bring you  $n$  dollars. A *Standard Assumption* in decision theory is that if the expected value of playing a game is  $n$ , then, for any  $m < n$ , you should be willing to pay \$ $m$  for the privilege of playing the game.

*Example:* Consider a game consisting in the toss of a fair coin. If the coin lands Heads, you get \$2, if it lands Tails you get \$-1 (i.e. you loose \$1). The expected

value of playing this game is  $1/2 \cdot 2 + 1/2 \cdot -1 = 0.5$ . So, according to the Standard Assumption, you should be willing to pay any amount less than \$0.5 for the privilege of playing this game.

The following problems are worth 10 points each:

- (a) A simplified version of the *Mega Millions* Lottery works as follows. The player selects five different numbers between 1 and 56 (inclusive). Later, 5 balls are drawn at random from an urn containing 56 balls, labelled 1 through 56. (Balls are not put back into the urn between drawings) If the 5 balls drawn from the urn match the numbers selected by the player, the player wins \$250,000. (Order doesn't matter.) Otherwise the player wins nothing. According to the Standard Assumption, how much should one be willing to pay for the privilege of playing this game. (*Fun fact:* a *Mega Millions* ticket currently costs \$1.)
- (b) According to the Standard Assumption, how much should one be willing to pay for the privilege of playing the game in problem 2(c)? Is this the right result? If so, explain why. If not, explain what is wrong with the standard assumption.
4. *Extra Credit:* Game Master has two envelopes. She decides how much money to put in the first envelope by rolling a fair (six-sided) die until it lands on a 1 or a 2. (If the die never lands on a 1 or a 2, the game is off.) If the die lands on a 1 or 2 on the first roll, Game Master fills the envelope with \$1; if the die lands on a 1 or a 2 on the second roll, Game Master fills the envelope with \$2, and, in general, if the die lands on a 1 or a 2 on the  $n$ th roll, Game Master fills the envelope with  $\$2^{n-1}$ . Game Master then puts twice as much money in the second envelope. You are offered one of the two envelopes without being told which is which. You pick one of them, and find that it contains  $\$2^n$  for some  $n$ . You are then offered a choice: you may keep the money in the envelope you selected, or give up the money and instead keep whatever money is in the other envelope. (You are not allowed to open the other envelope before making your choice.)

According to the Standard Assumption, should you switch? If the answer is 'no', explain why not. If the answer is 'yes', explain why you shouldn't have picked the envelope you will ultimately switch to from the start, instead of picking the wrong envelope first, and then switching.

(10 points.)

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24.118 Paradox & Infinity  
Spring 2013

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