ESD: Recitation #5

The barbershop revisited

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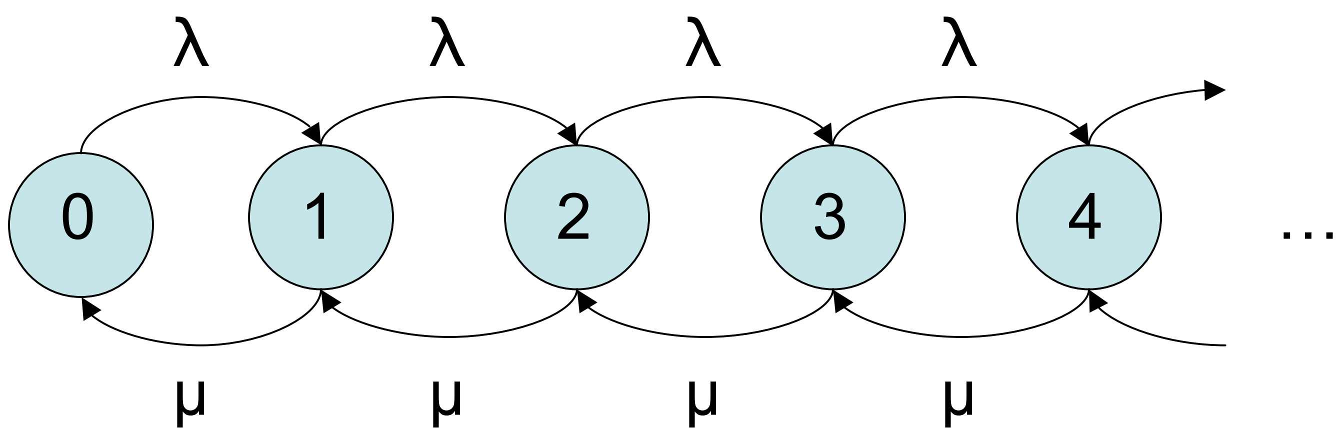
Infinite number of waiting seats

- One barber, infinite number of chairs for waiting customers.
- Prospective customers arrive in a Poisson manner at the rate of λ per hour.
- It takes the barber $1/\mu$ on average to serve a customer ($\lambda = 0.9 \times \mu$).
- No prospective customer is ever lost.
- What is the average number of customers?

Model



Solving (1)

- What is the probability that N customers are in the barbershop?

$$P_1 = \frac{\lambda}{\mu} P_0; P_2 = \frac{\lambda}{\mu} P_1$$

$$P_N = \left(\frac{\lambda}{\mu}\right)^N P_0$$

$$\sum_{i=0}^{\infty} P_i = 1 \Rightarrow P_0 = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k} = \frac{1}{\sum_{k=0}^{\infty} 0.9^k} = \frac{1}{10} = 0.1$$

Solving (2)

- Average number of customers:

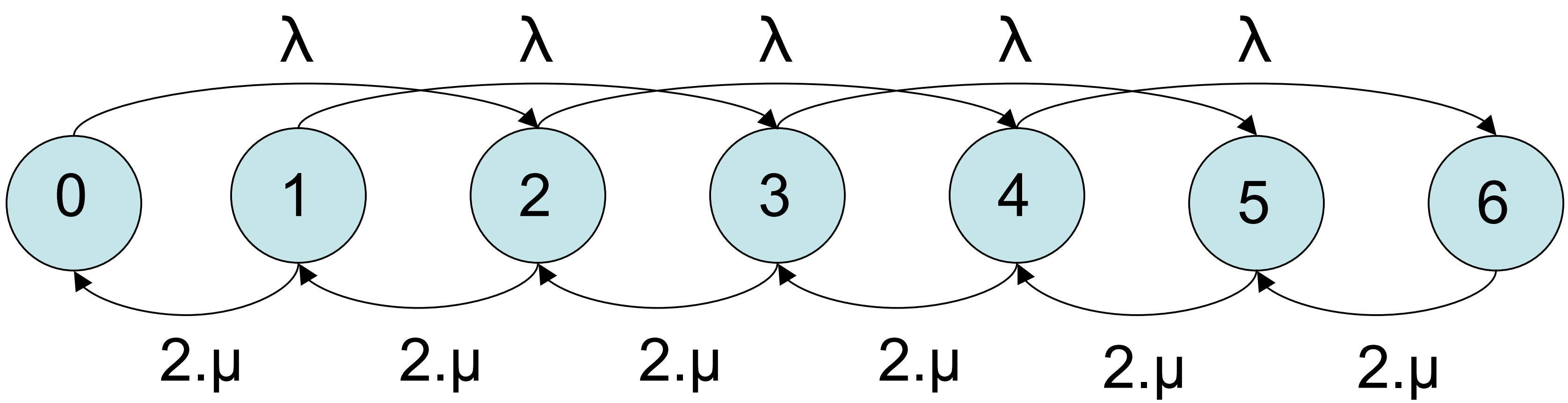
$$P_N = 0.1 \times \left(\frac{\lambda}{\mu}\right)^N = 0.1 \times 0.9^N$$

$$E[Nb\_customers] = \sum_{k=0}^{\infty} k.P_k = 0.1 \times \sum_{k=0}^{\infty} k \times 0.9^k = 9$$

Different service completion rate

- One barber, two chairs for waiting customers.
- Prospective customers arrive in a Poisson manner at the rate of λ per hour.
- It takes the barber $1/\mu$ on average to serve a customer. The service completion rate is described by a second order Erlang pdf. Assume $\lambda = \mu$.
- Prospective customers finding the barbershop full are lost forever.
- What is the average number of customers?

Model



Arrival rate: $f_A(x) = \lambda . e^{-\lambda . x}, x \geq 0$

Service rate: $f_S(x) = 4.\mu^2.x.e^{-2.\mu.x}, x \geq 0$

Solving (1)

- Steady-state probabilities:

$$\lambda.P_0 = 2\mu.P_1$$

$$(2\mu + \lambda).P_1 = 2\mu.P_2$$

$$(2\mu + \lambda).P_1 = 2\mu.P_2$$

$$(2\mu + \lambda).P_2 = \lambda.P_0 + 2\mu.P_3$$

$$(2\mu + \lambda).P_3 = \lambda.P_1 + 2\mu.P_4$$

$$(2\mu + \lambda).P_4 = \lambda.P_2 + 2\mu.P_5$$

$$2\mu.P_5 = \lambda.P_3 + 2\mu.P_6$$

$$2\mu.P_6 = \lambda.P_4$$

Solving (2)

- Calculate P_0:

$$\sum_{k=0}^{6} P_k = 1 \Leftrightarrow P_0 + \frac{1}{2}P_0 + \frac{3}{4}P_0 + \frac{5}{8}P_0 + \frac{11}{16}P_0 + \frac{21}{32}P_0 + \frac{11}{32}P_0 = \frac{65}{16}P_0$$

$$\sum_{k=0}^{6} P_k = 1 \Leftrightarrow P_0 = \frac{16}{65}$$

Solving (3)

- Average number of customers:

$$E[Nb\_customers] = 0 \times \frac{16}{65} + 1 \times \left(\frac{8}{65} + \frac{12}{65} \right)$$

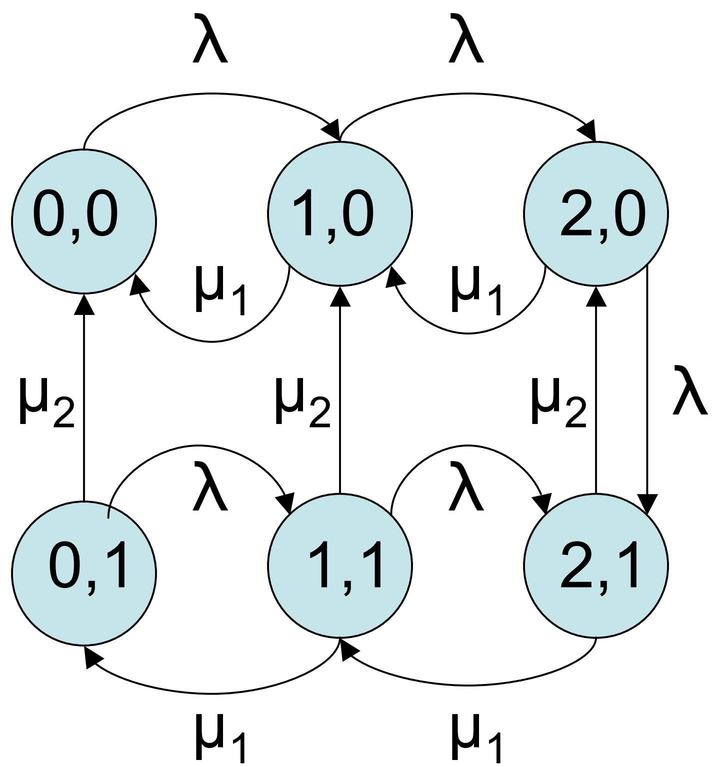
$$+2 \times \left(\frac{2}{13} + \frac{11}{65} \right) + 3 \times \left(\frac{21}{130} + \frac{11}{130} \right)$$

$$E[Nb\_customers] = \frac{22}{13} \approx 1.692$$

Additional barber

- Two barbers:
  - Adam (takes $1/\mu_1$ on average to serve a customer)
  - Ben (takes $1/\mu_2$ on average to serve a customer)
- One chair for waiting customers.
- Prospective customers arrive in a Poisson manner at the rate of λ per hour.
- Prospective customers finding the barbershop full are lost forever

Modeling the system



State of the system: $S_{i,j}$

i: number of people being serviced by or waiting for Adam

j: number of people being serviced by Ben

The question

- Suppose $\lambda = \mu_1 = \mu_2$.
- What is the probability that Ben is busy at a random time?

Solving (1)

- Steady-state probabilities:

$$\lambda.P_{0,0} = \mu_1.P_{1,0} + \mu_2.P_{0,1}$$

$$(\mu_1 + \lambda).P_{1,0} = \lambda.P_{0,0} + \mu_1.P_{2,0} + \mu_2.P_{1,1}$$

$$(\mu_1 + \lambda).P_{2,0} = \lambda.P_{1,0} + \mu_2.P_{2,1}$$

$$(\mu_1 + \mu_2).P_{2,1} = \lambda.P_{2,0} + \lambda.P_{1,1}$$

$$(\mu_1 + \mu_2 + \lambda).P_{1,1} = \lambda.P_{0,1} + \mu_1.P_{2,1}$$

$$(\mu_2 + \lambda).P_{0,1} = \mu_1.P_{1,1}$$

Solving (2)

- P{Ben busy} = $P_{0,1} + P_{1,1} + P_{2,1}$
- Using:

$$P_{0,0} + P_{1,0} + P_{2,0} + P_{2,1} + P_{1,1} + P_{0,1} = 1$$

- We find:

$$P\{Ben\_busy\} = \frac{8}{129} + \frac{4}{43} + \frac{22}{129} = \frac{42}{129} \approx 0.326$$