ESD: Recitation #4

Birthday problem
An approximate method

- Bernoulli trials
- Number of trials to compare birthdays of all people in the class:

$$N = \frac{n!}{(n-2)!\,2!} = \frac{n!}{2(n-2)!}$$

- Probability that nobody has the same birthday than someone else:

$$P_0 = \binom{N}{0}\left(\frac{1}{365}\right)^0\left(\frac{364}{365}\right)^N = \left(\frac{364}{365}\right)^N = \left(\frac{364}{365}\right)^{\frac{n!}{2(n-2)!}}$$

The exact solution

- Probability that nobody has the same birthday than anybody else:

$$P_0 = \prod_{i=0}^{n-1}\left(1 - \frac{i}{365}\right) = \frac{365!}{365^n (365-n)!}$$

What was the average travel distance between two random points in Budapest in the 1850s?

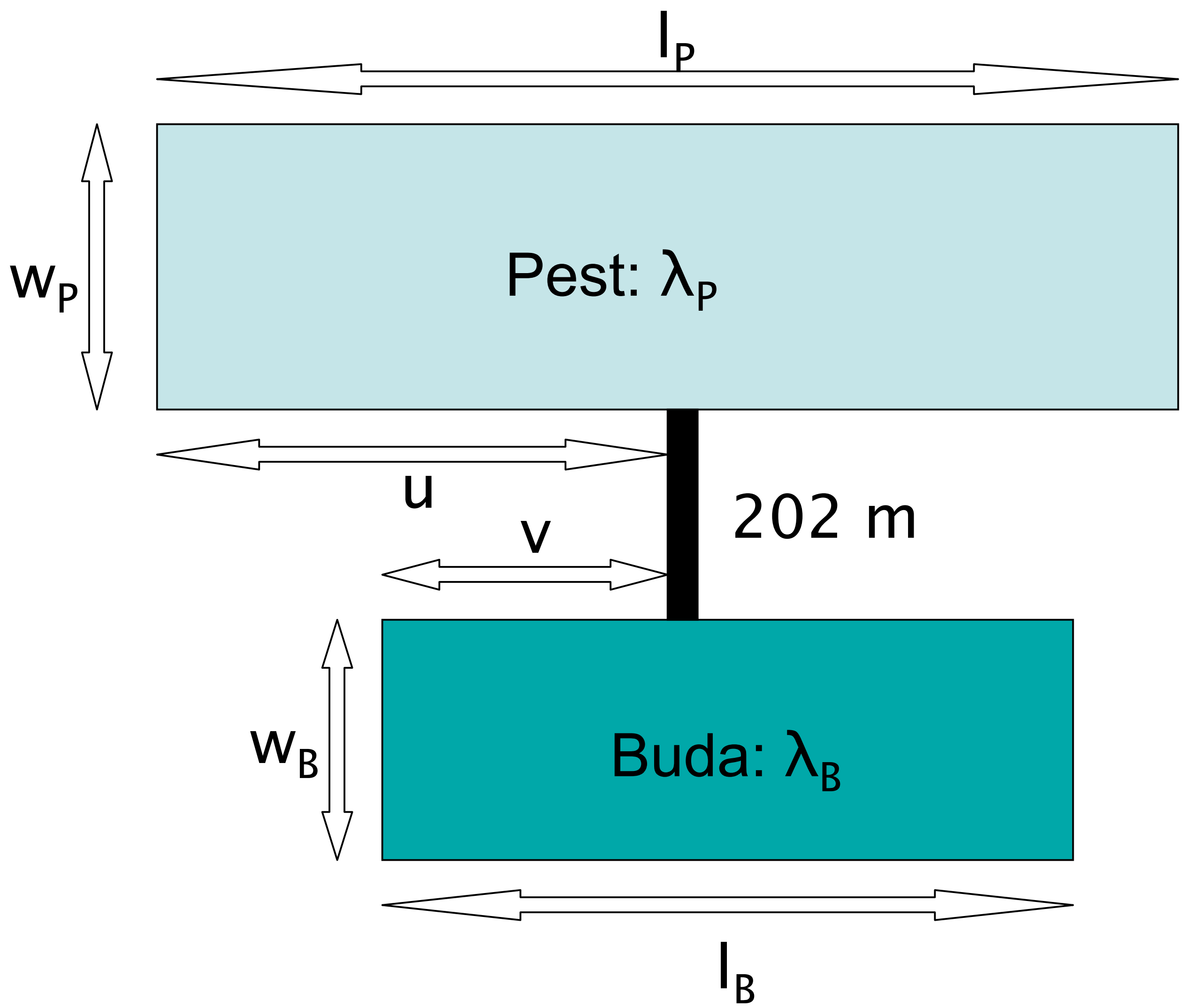
Budapest = Buda + Pest

Photo removed due to copyright restrictions.
The Danube River through Budapest, showing the two shores.

Only one bridge: Széchenyi Lánchid (Chain bridge)

Source: Wikipedia

Modeling



Within each city

- In Buda:

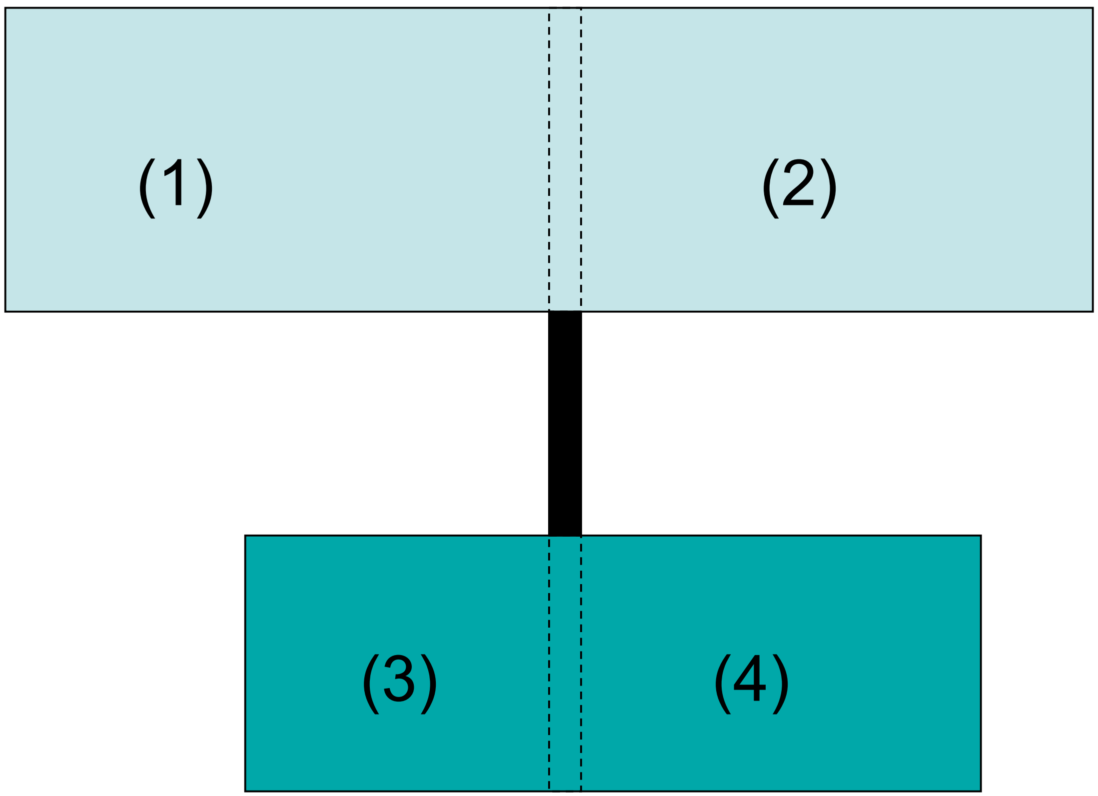
$$P_{B-B} = \left(\frac{w_B . l_B . \lambda_B}{w_B . l_B . \lambda_B + w_P . l_P . \lambda_P} \right)^2 \qquad \overline{D_B} = \frac{1}{3}(w_B + l_B)$$

- In Pest:

$$P_{P-P} = \left(\frac{w_P . l_P . \lambda_P}{w_B . l_B . \lambda_B + w_P . l_P . \lambda_P} \right)^2 \qquad \overline{D_P} = \frac{1}{3}(w_P + l_P)$$

Between the two cities

- 4 cases:



Between (1) and (3)

- Probability:

$$P_{(1)-(3)} = 2\frac{w_P.l_P.\lambda_P.w_B.l_B.\lambda_B}{\left(w_B.l_B.\lambda_B + w_P.l_P.\lambda_P\right)^2} \times \frac{u.v}{l_P.l_B}$$

- Average Distance:

$$\overline{D_{(1)-(3)}} = \frac{1}{2}(w_B + v) + \frac{1}{2}(w_P + u) + 202$$

Between (1) and (4)

- Probability:

$$P_{(1)-(4)} = 2\frac{w_P.l_P.\lambda_P.w_B.l_B.\lambda_B}{\left(w_B.l_B.\lambda_B + w_P.l_P.\lambda_P\right)^2} \times \frac{u.(l_B - v)}{l_P.l_B}$$

- Average Distance:

$$\overline{D_{(1)-(4)}} = \frac{1}{2}(w_P + u) + \frac{1}{2}(w_B + (l_B - v)) + 202$$

And continue…

- Between (2) and (3)
- Between (2) and (4)

- Get the final answer…

More complications

- There is currently ten bridges on the Danube.

- How does average traveling distance change if we build another one?

Bertrand’s Paradox

Joseph Louis François Bertrand

(1822-1900)

Wrote *Calcul des probabilités* in 1888.

The question

- Consider an equilateral triangle inscribed in a circle. Suppose a cord of the circle is chosen at random.
- What is the probability that the chord is longer than a side of the triangle?

Random endpoints

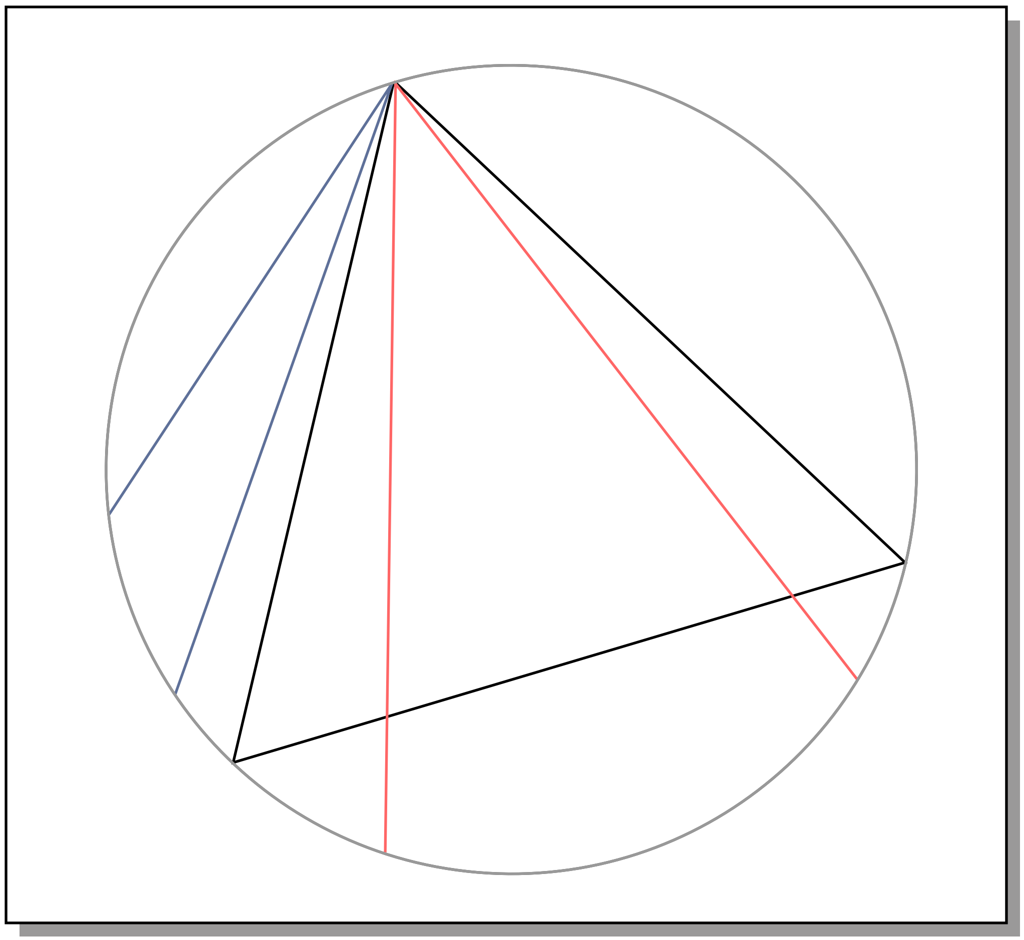


Figure by MIT OCW.

Random radius

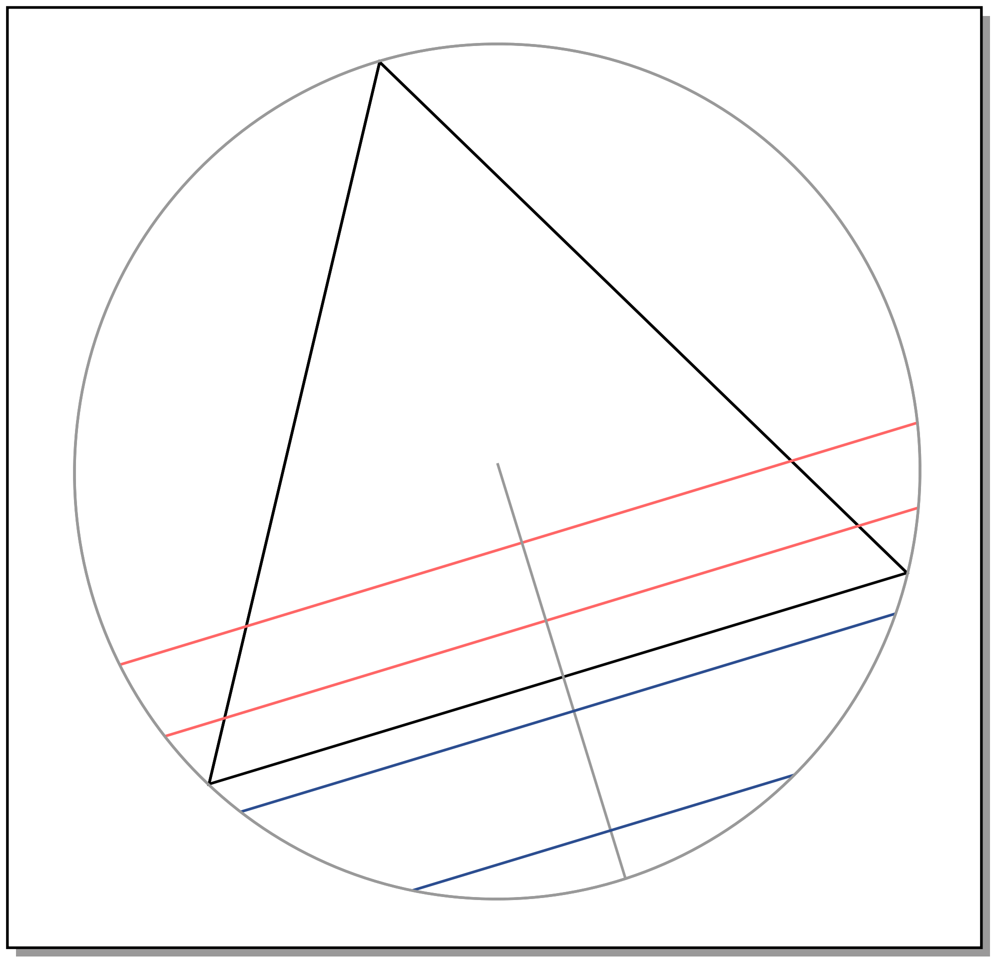


Figure by MIT OCW.

Random midpoints

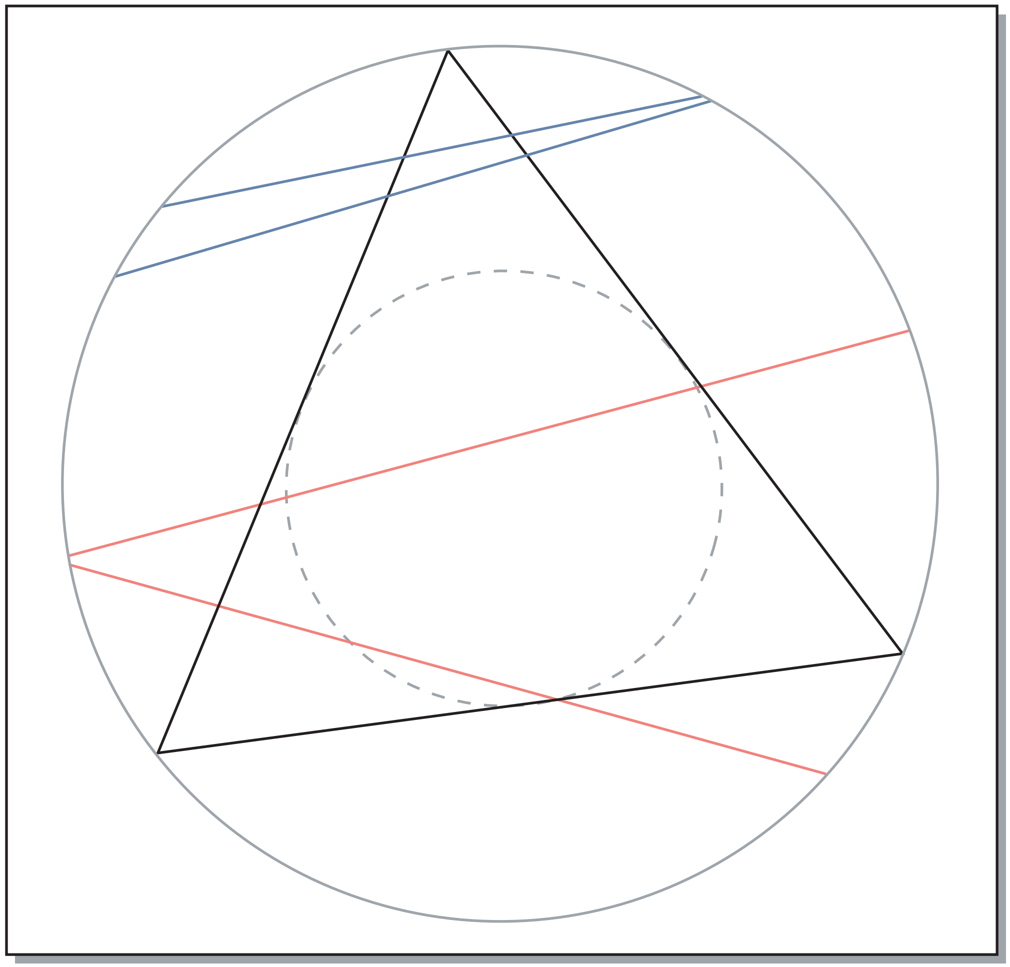
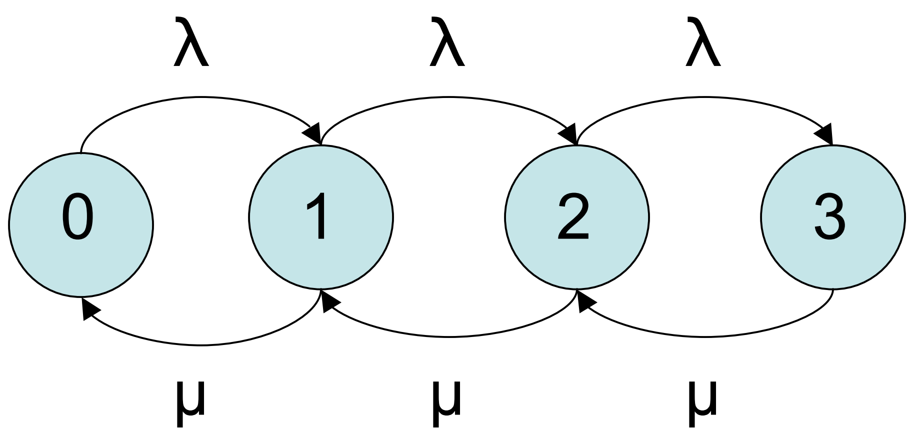


Figure by MIT OCW.

Barbershop

- One barber, two chairs for waiting customers.
- Prospective customers arrive in a Poisson manner at the rate of λ per hour.
- It takes the barber $1/\mu$ on average to serve a customer.
- Prospective customers finding the barbershop full are lost forever.
- What is the average number of customers?

Model



$$\lambda = 2.\mu$$

Solving (1)

- What is the probability that N customers are in the barbershop?

$$P_1 = \frac{\lambda}{\mu} P_0 ; P_2 = \frac{\lambda}{\mu} P_1$$

$$P_N = \left(\frac{\lambda}{\mu} \right)^N P_0$$

$$\sum_{i=0}^{3} P_i = 1 \Rightarrow P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu} \right)^{3+1}}$$

Solving (2)

- Average number of customers:

$$P_N = \left(\frac{\lambda}{\mu}\right)^3 \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{3+1}} = \frac{2^3}{2^4-1}$$

$$E[Nb\_customers] = \sum_{i=0}^{3} i.P_i = \frac{34}{15} \approx 2.2667$$