ESD: Recitation #2

Definitions: 1. Expectation

- Expectation (or population mean): E[X] or μ

$$\mathrm{E}(X) = \sum_i p_i x_i$$

- Linearity: E[aX + b] = a.E[X] + b
- Non-multiplicativity: E[X.Y] = E[X].E[Y] iff X and Y are independently distributed

Definitions: 2. Median

- Median: c
- The median of a probability distribution is the value of c that minimizes

$$E(|X - c|)$$

- The median is not necessarily unique.

Definitions: 3. Variance

- Variance: var[X] or σ^2

$$\mathrm{var}[X] = E[(X-\mu)^2] = E(X - E[X])^2]$$

$$\mathrm{var}[X] = E[X^2] - (E[X])^2$$

- Non-linearity:

$$\mathrm{var}(aX + bY) = a^2\mathrm{var}(X) + b^2\mathrm{var}(Y) + 2ab\,\mathrm{cov}(X,Y).$$

The Democrates in the US Senate

- 106th Congress: 45
- 107th Congress: 50
- 108th Congress: 48
- 109th Congress: 45
- 110th Congress: 51

Return to the geometric distribution

- Calculating the expectation:

$$E[x] = \sum_{n=0}^{\infty} np(1-p)^n = ?$$

$$\sum_{n=0}^{m} x^n = \frac{1-x^{n+1}}{1-x}$$

$$\frac{d}{dx}\left(\sum_{n=0}^{m} x^n\right) = \sum_{n=0}^{m} n.x^{n-1} = \frac{-(n+1).x^n(1-x)+(1-x^{n+1}).1}{(1-x)^2}$$

- For the variance calculation, cf.: http://www.win.tue.nl/~rnunez/2DI30/NoteOnGeometricDistribution/distributions.pdf

z-Transform

z-transform of $p_X(x)$:

$$p_X^T(z) \equiv E[z^X] = \sum_{x=0}^{\infty} z^x p_X(x)$$

Why is it useful? You can use it to determine the expectation and variance of probability distributions:

$$E[X] = \left[\frac{dp_X^T(z)}{dz}\right]_{z=1}$$

$$E[X^2] = \left[\frac{d^2 p_X^T(z)}{dz^2}\right]_{z=1} + \left[\frac{dp_X^T(z)}{dz}\right]_{z=1}$$

Z-Transform (cont'd)

- For the most common z-transforms, have a look at this website:

http://www.swarthmore.edu/NatSci/echeeve1/Ref/LPSA/LaplaceZTable/LaplaceZFuncTable.html

Buffon’s Needle

Georges-Louis Leclerc,
Comte de Buffon
(1707-1788)

Source: Wikipedia

Probability treatise
Sur le jeu de franc-carreau

The Experiment

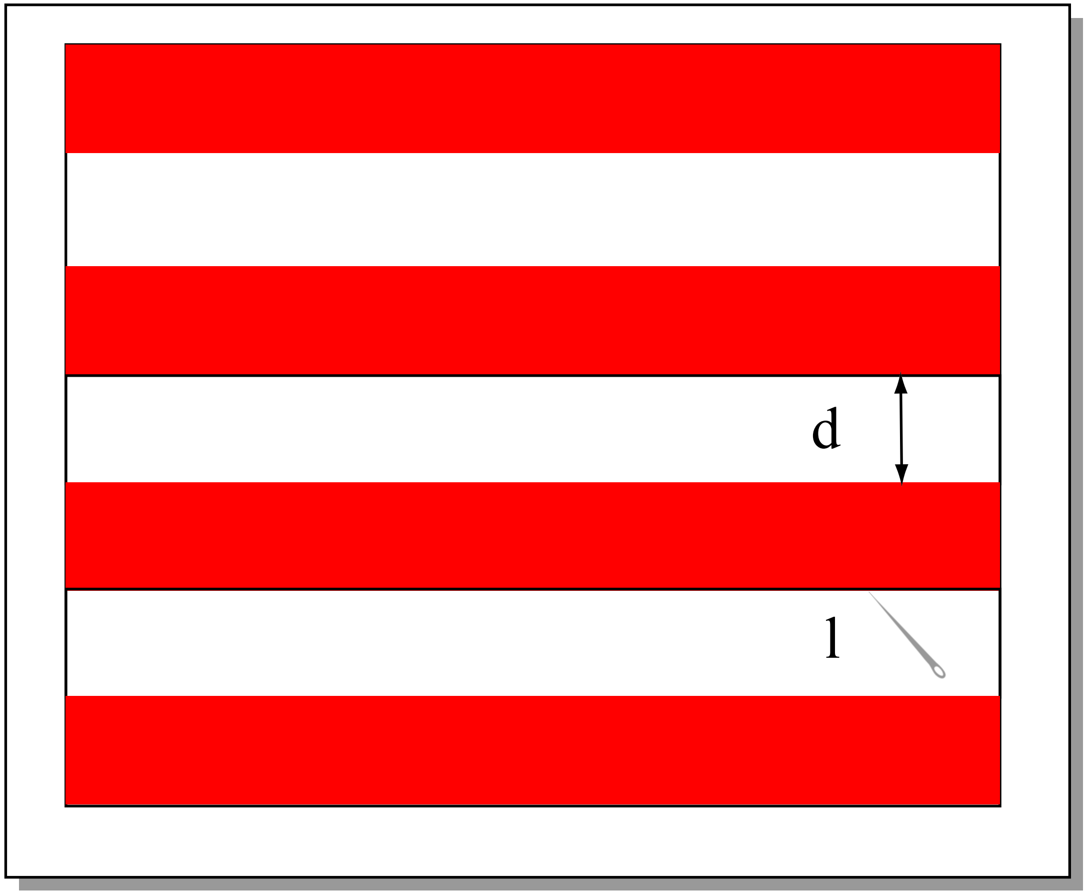


Figure by MIT OCW.

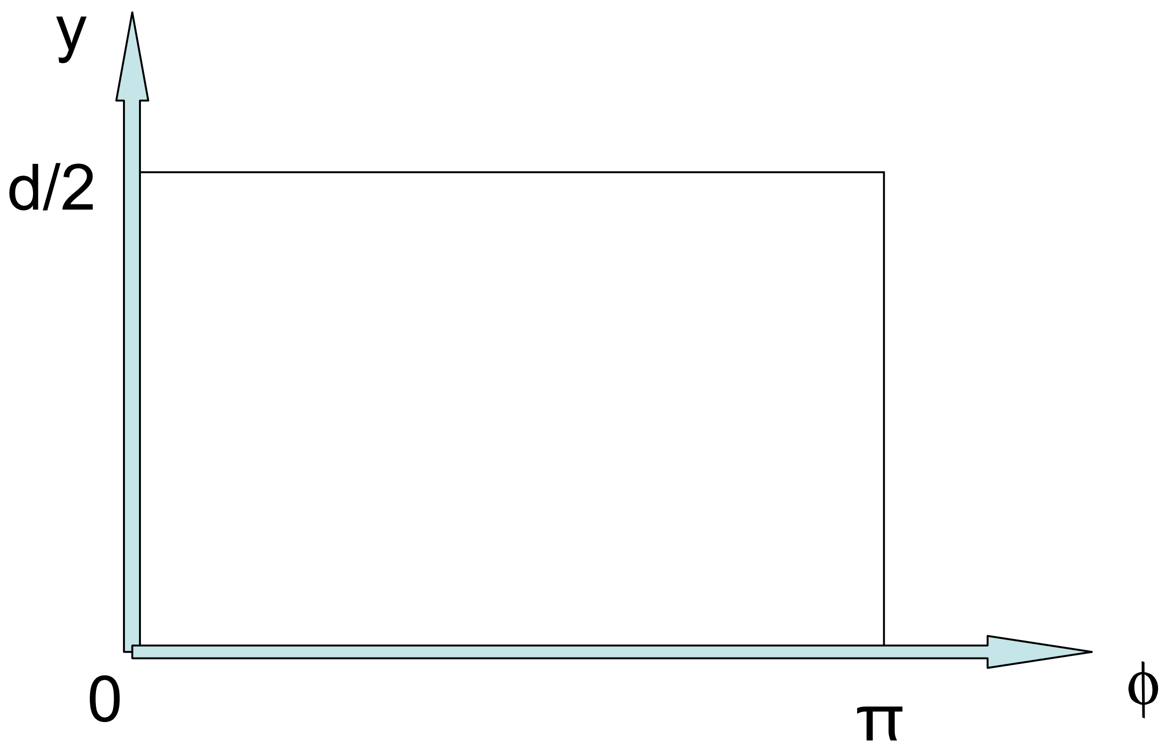
$l \leq d$

Random variables:

ϕ = angle between
needle and stripe
interface

y = distance between needle middle and
nearest stripe interface

Joint sample space



Joint probability distribution

- Determine $f_{Y,\phi}(y,\phi)$
- $f_Y(y)$ and $f_\phi(\phi)$ independent, uniformly distributed
- $f_Y(y)$ = constant = 2/d
- $f_\phi(\phi)$ = constant = $1/\pi$
- $f_{Y,\phi}(y,\phi) = f_Y(y) \times f_\phi(\phi) = 2/\pi d$

Working in the joint sample space

- Needle crosses stripe interface <=>

  [y - (l/2)sinϕ] < 0 or y < (l/2)sinϕ

$$P = \int_0^{\phi} \left(\int_0^{\frac{l}{2}\sin\phi} \frac{2}{\pi d}\, dy \right) d\phi = \frac{l}{\pi d} \int_0^{\phi} \sin\phi . d\phi = \frac{2l}{\pi d}$$