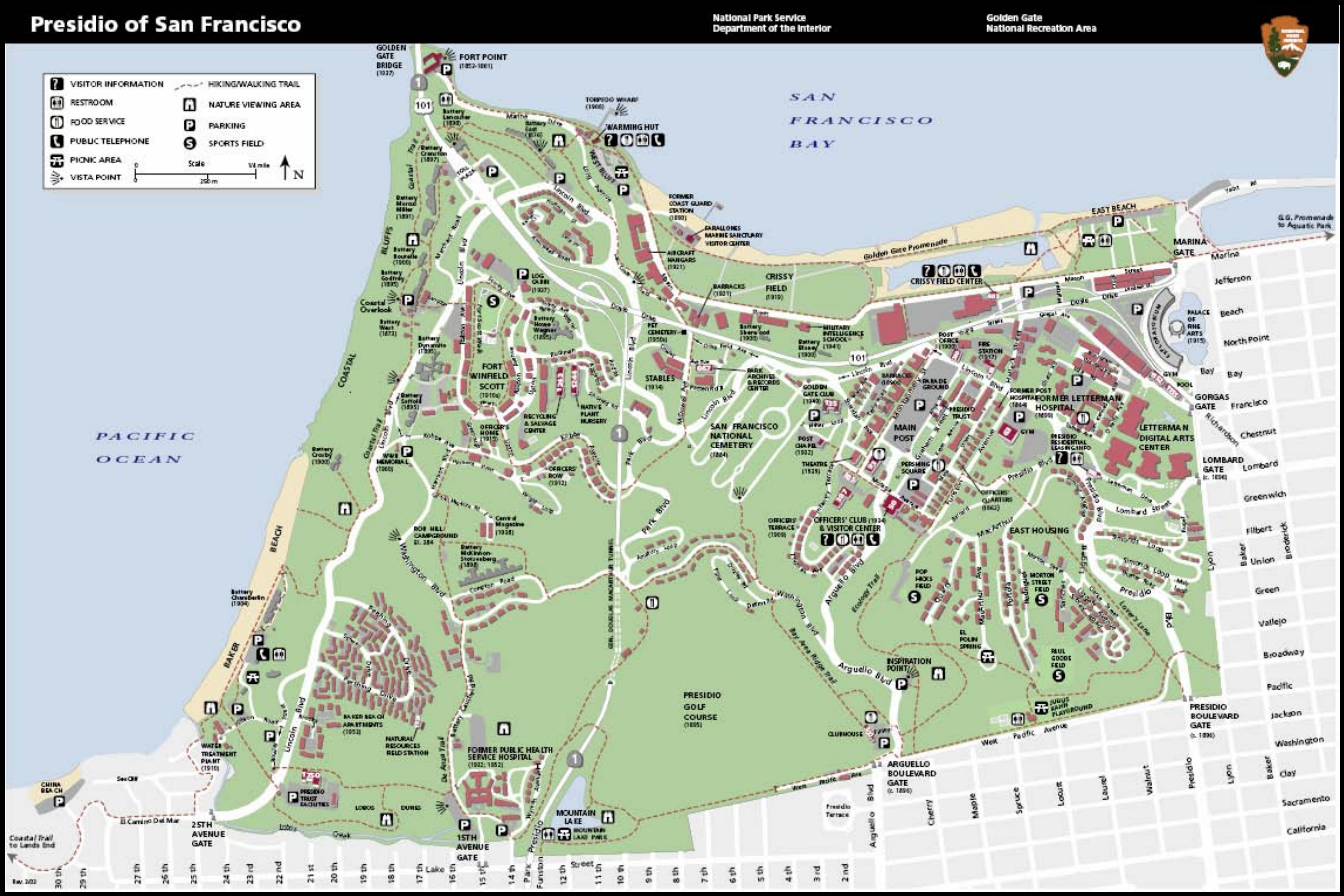
ESD.86 Spatial Models



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Outline

- Min, Max
- Ratio of Urban Distance to Airplane Dist.
- Spatial Poisson Processes
- Facility Location

Min, Max. Deriving a Joint PDF

Suppose X_1 and X_2 are i.i.d. uniformly distributed over [0, 1]. They could, for instance, be the locations of 2 police cars. We seek to derive the joint pdf of Y and Z, where

$$Y = \text{Min}(X_1, X_2)$$
$$Z = \text{Max}(X_1, X_2)$$

That is, we seek

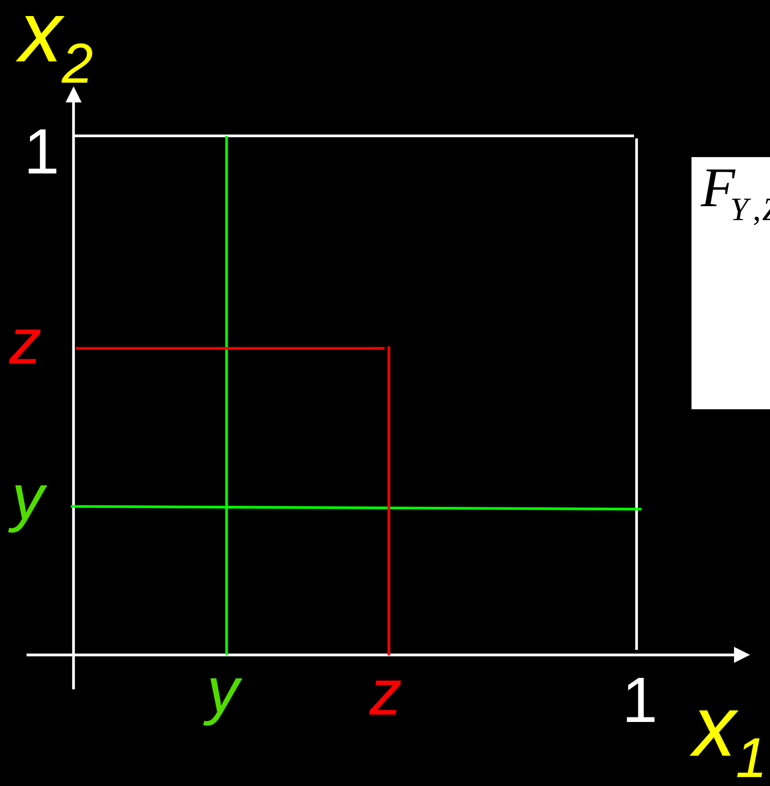
$$f_{Y,Z}(y,z)\,dy\,dz = P\{y < Y < y+dy, z < Z < z+dz\}$$

Four Steps to Happiness

1. Define the random variables
   Y=Min(X_1, X_2)
   Z=Max(X_1, X_2)
2. Define the joint sample space:
   Unit square in positive quadrant
3. Identify the probability measure over the joint sample space: **Uniform.**
4. Work within the sample space to answer any questions of interest.

http://www.schwimmerlegal.com/smiley.jpg

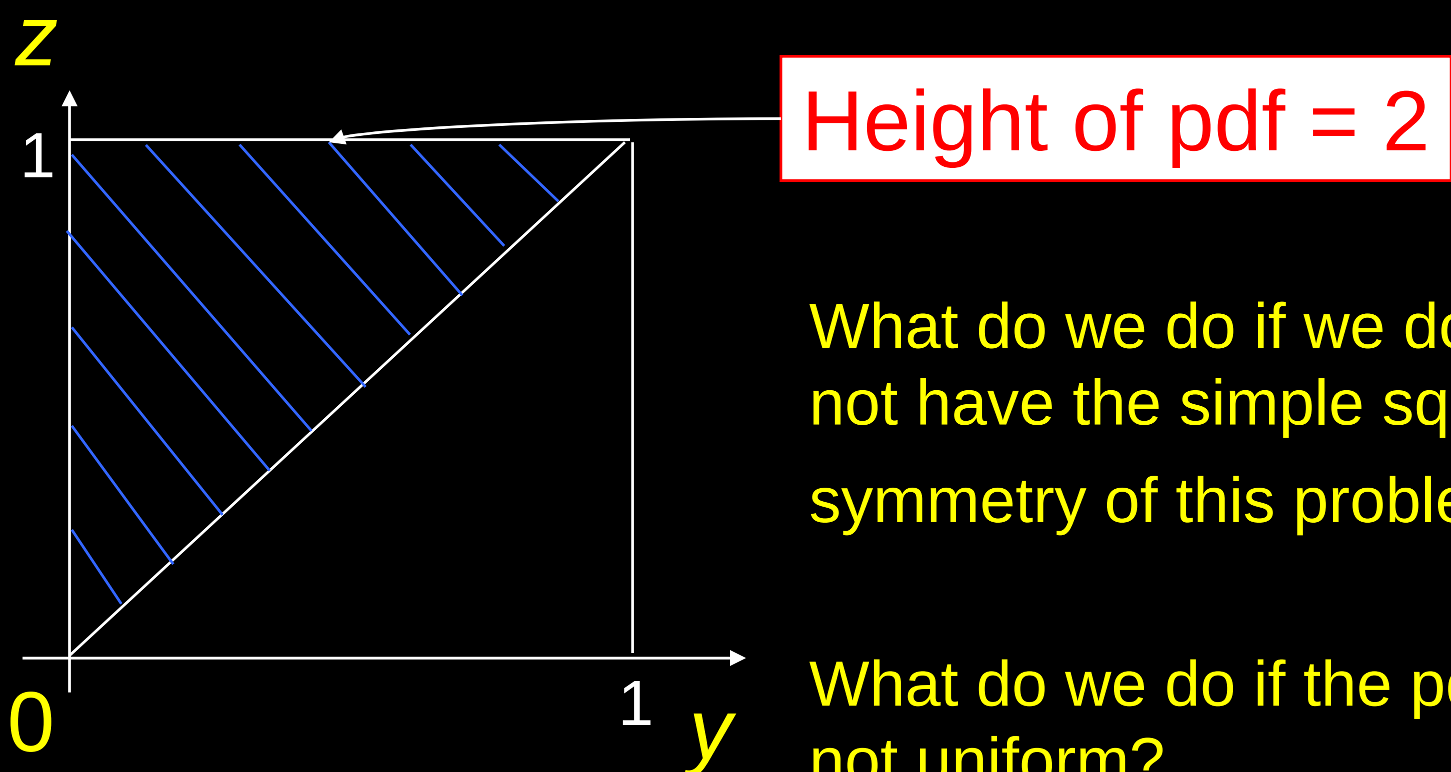
Work within the sample space



Use CDF

$$F_{Y,Z}(y,z) \equiv P\{Y \leq y, Z \leq z\}$$

Work within the sample space



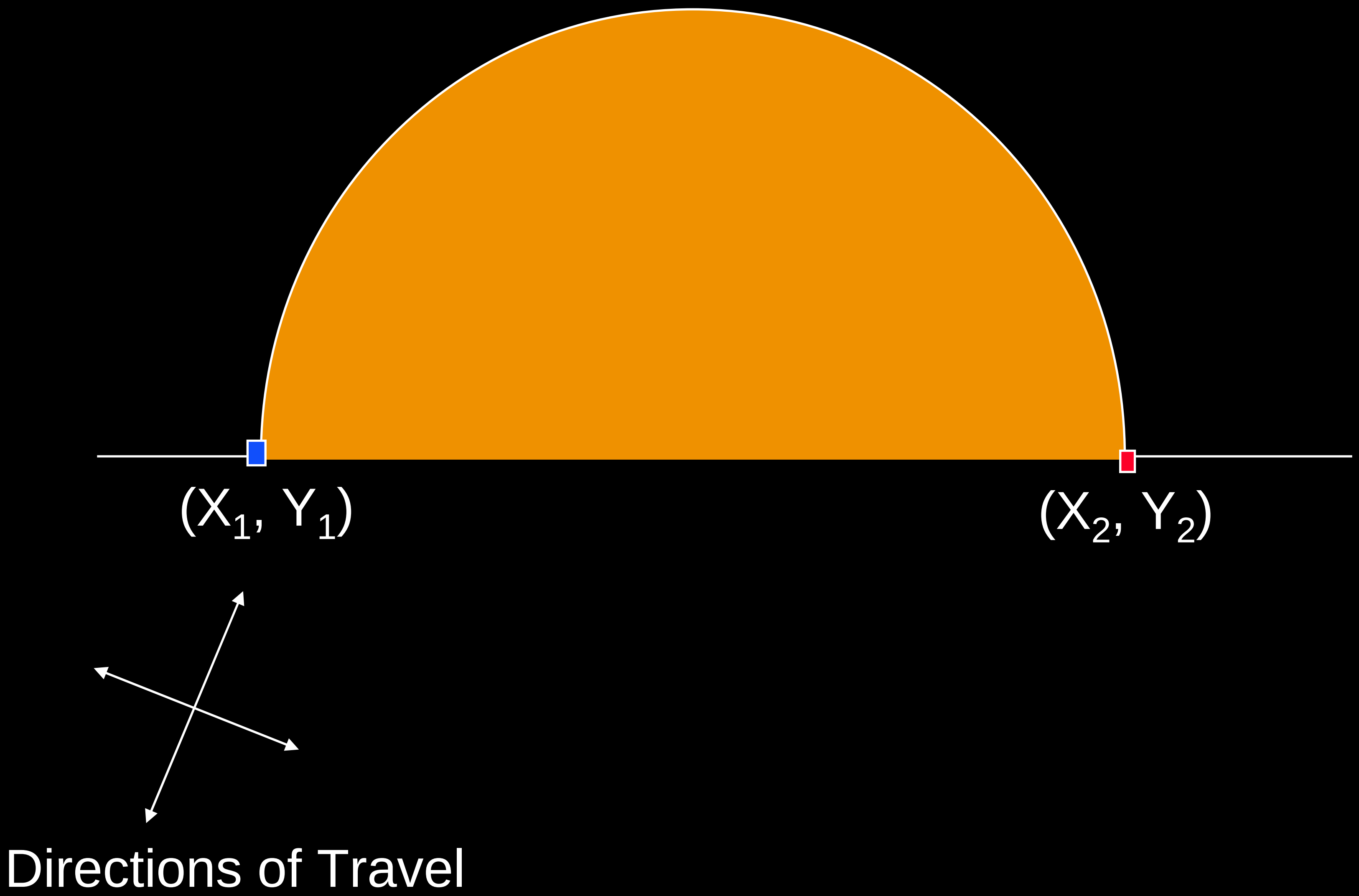
What do we do if we do not have the simple square symmetry of this problem?

What do we do if the pdf is not uniform?

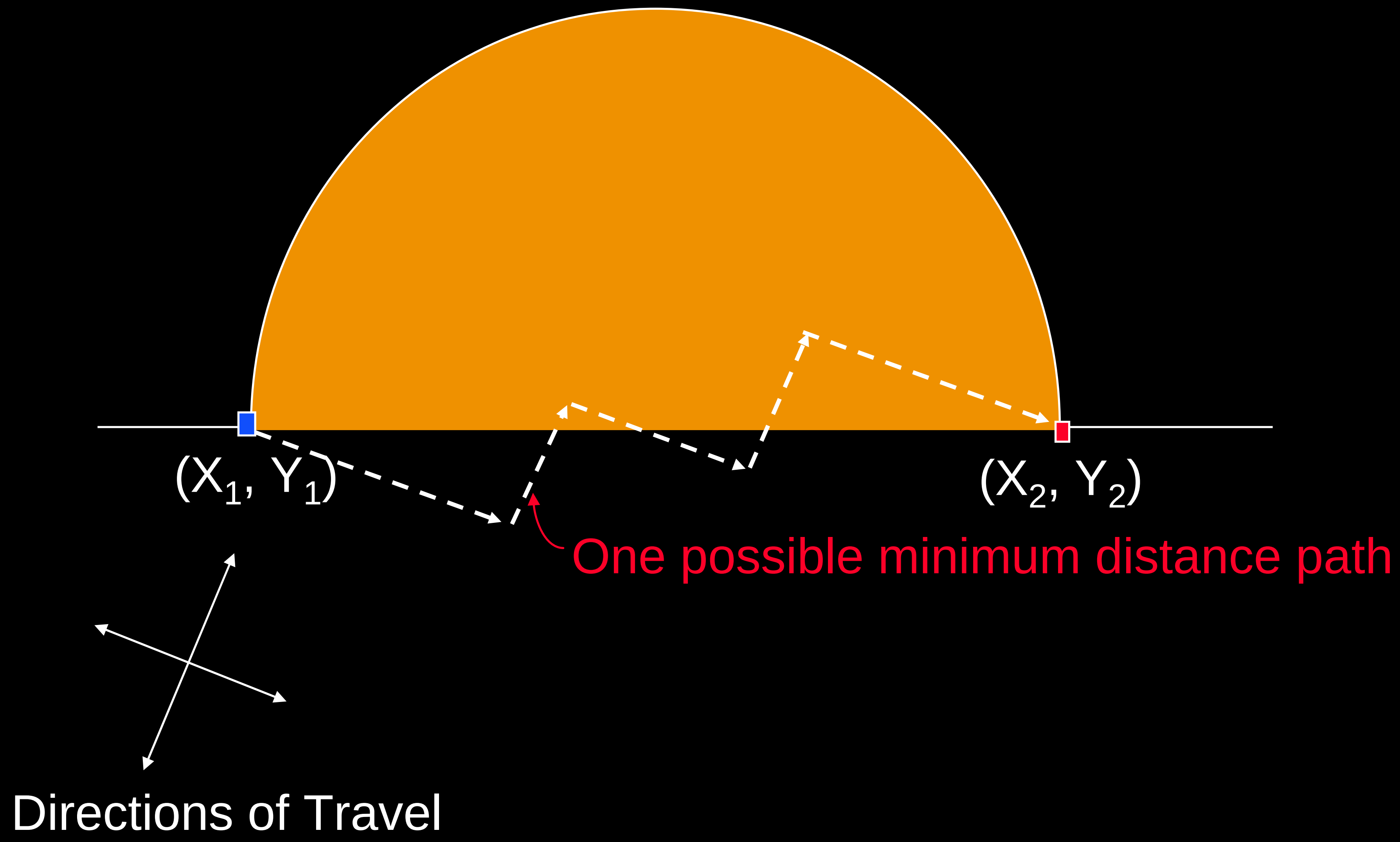
$$f_{Y,Z}(y,z) = \frac{\partial^2}{\partial y \partial z} F_{Y,Z}(y,z) = \frac{\partial}{\partial y}(2y) = 2,\ 0 \le y \le z \le 1$$

Ratio of Manhattan to Euclidean Distance Metrics

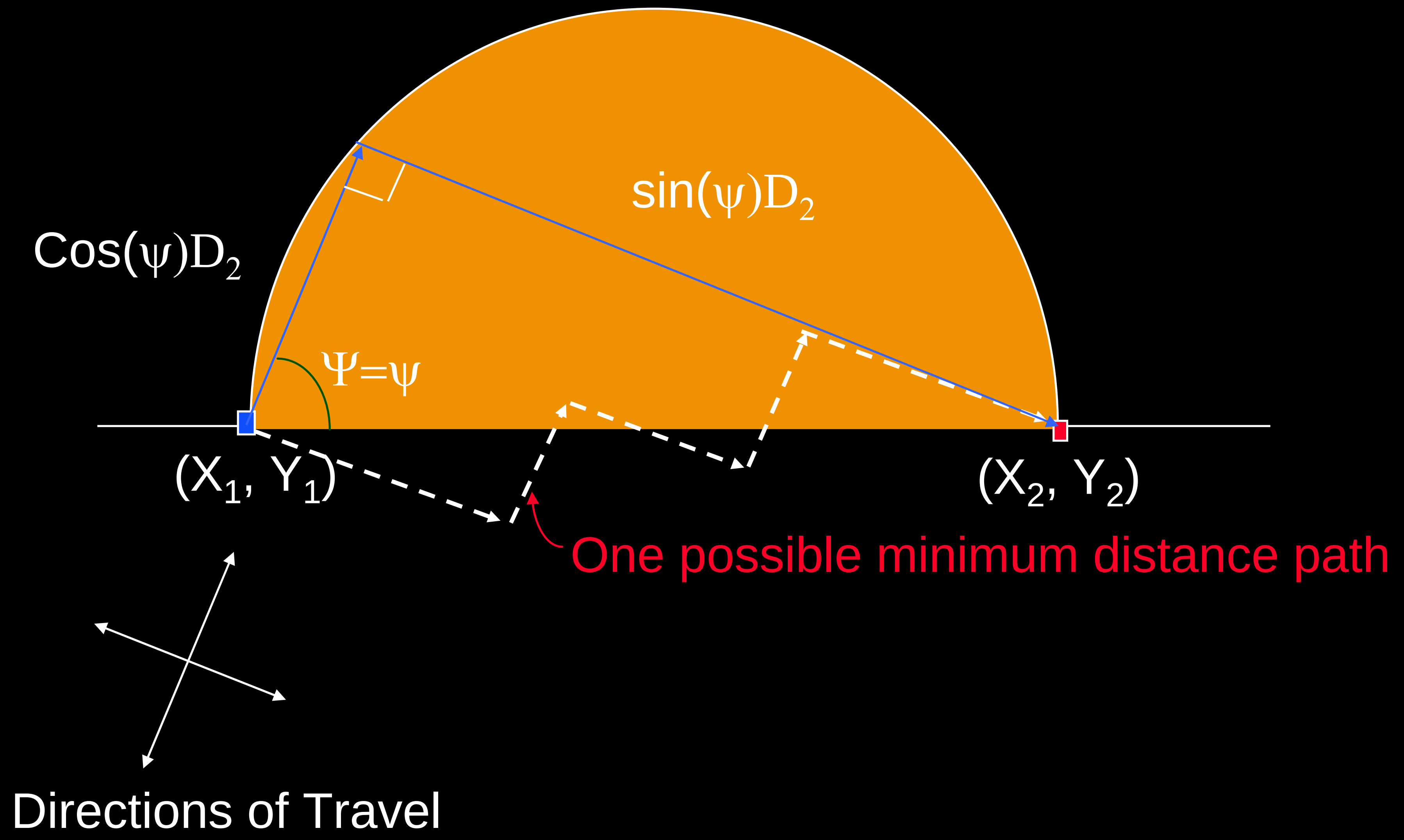
- 1. Define R.V.'s
  - » $D_1 = |X_1 - X_2| + |Y_1 - Y_2|$
  - » $D_2 = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$
  - » Ratio = R = D_1 / D_2
  - Ψ = angle of directions of travel wrt straight line connecting (X_1 , Y_1) & (X_2 , Y_2)



(X_1, Y_1)
(X_2, Y_2)
Directions of Travel



(X1, Y1)
(X2, Y2)
One possible minimum distance path
Directions of Travel

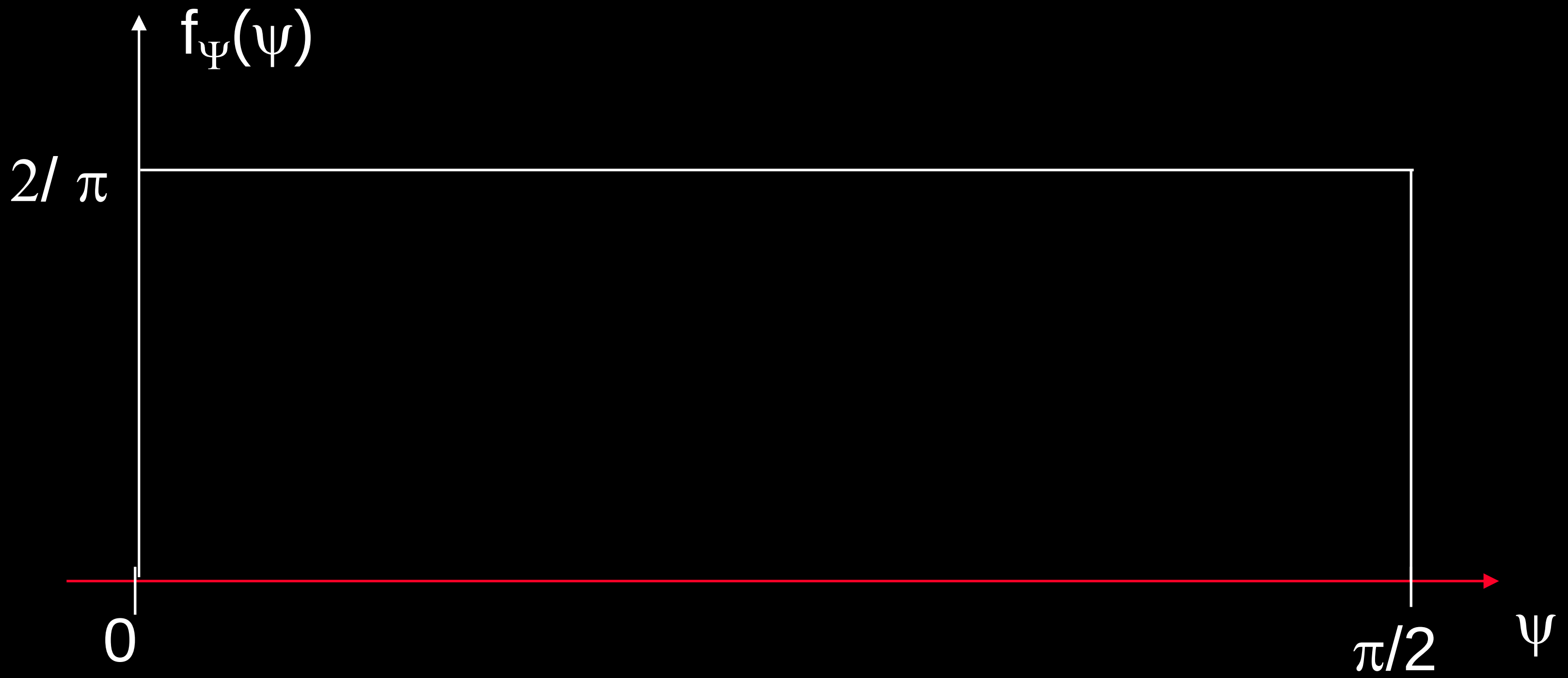

sin(ψ)D2
Cos(ψ)D2
Ψ=ψ
(X1, Y1)
(X2, Y2)
One possible minimum distance path
Directions of Travel

$$(R|\Psi) = \cos\Psi + \sin\Psi = 2^{1/2}\cos(\Psi - \pi/4)$$

2. Identify Sample Space

3. Probability Law over Sample Space:

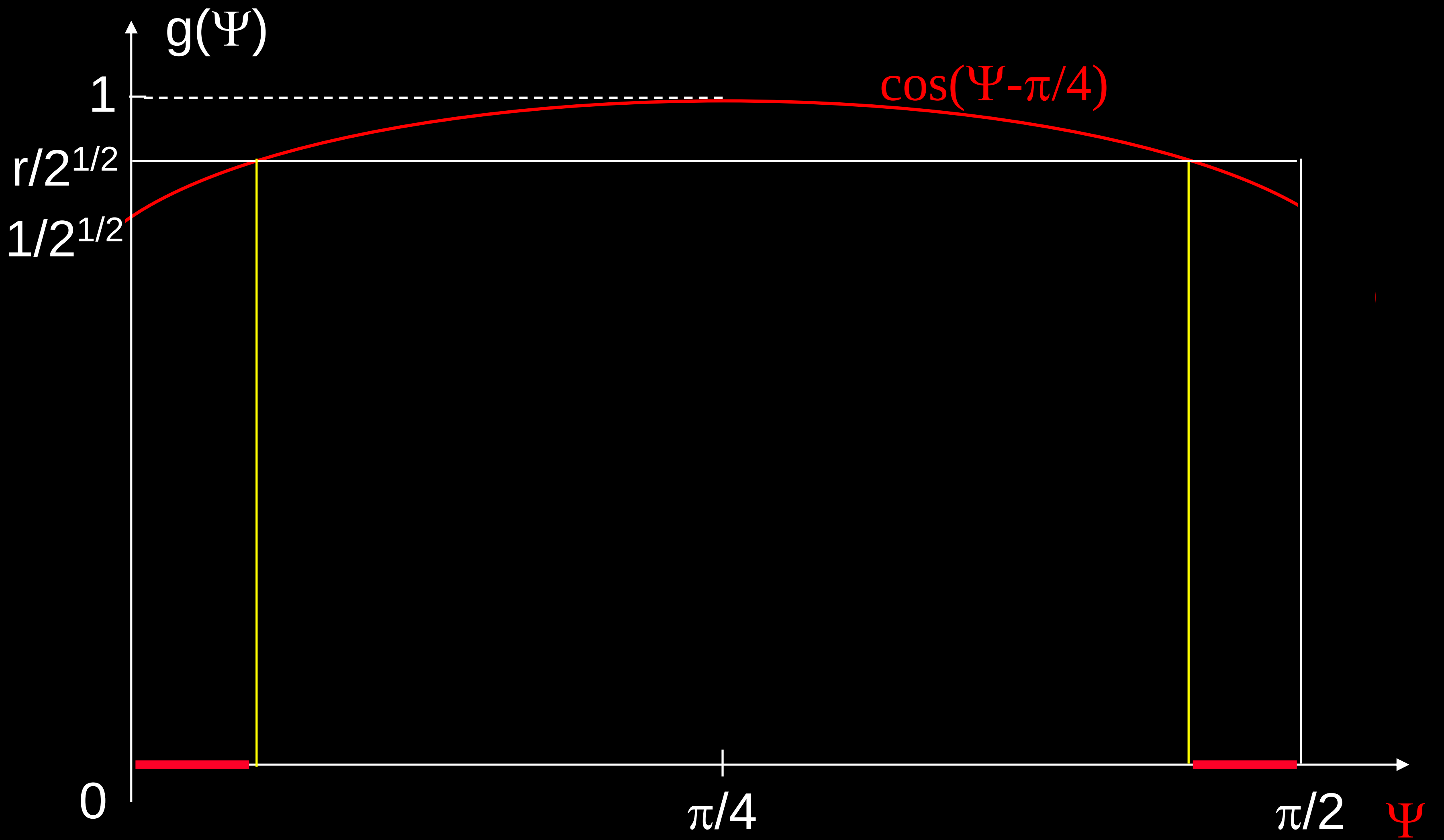
Invoke isotropy implying uniformity of angle

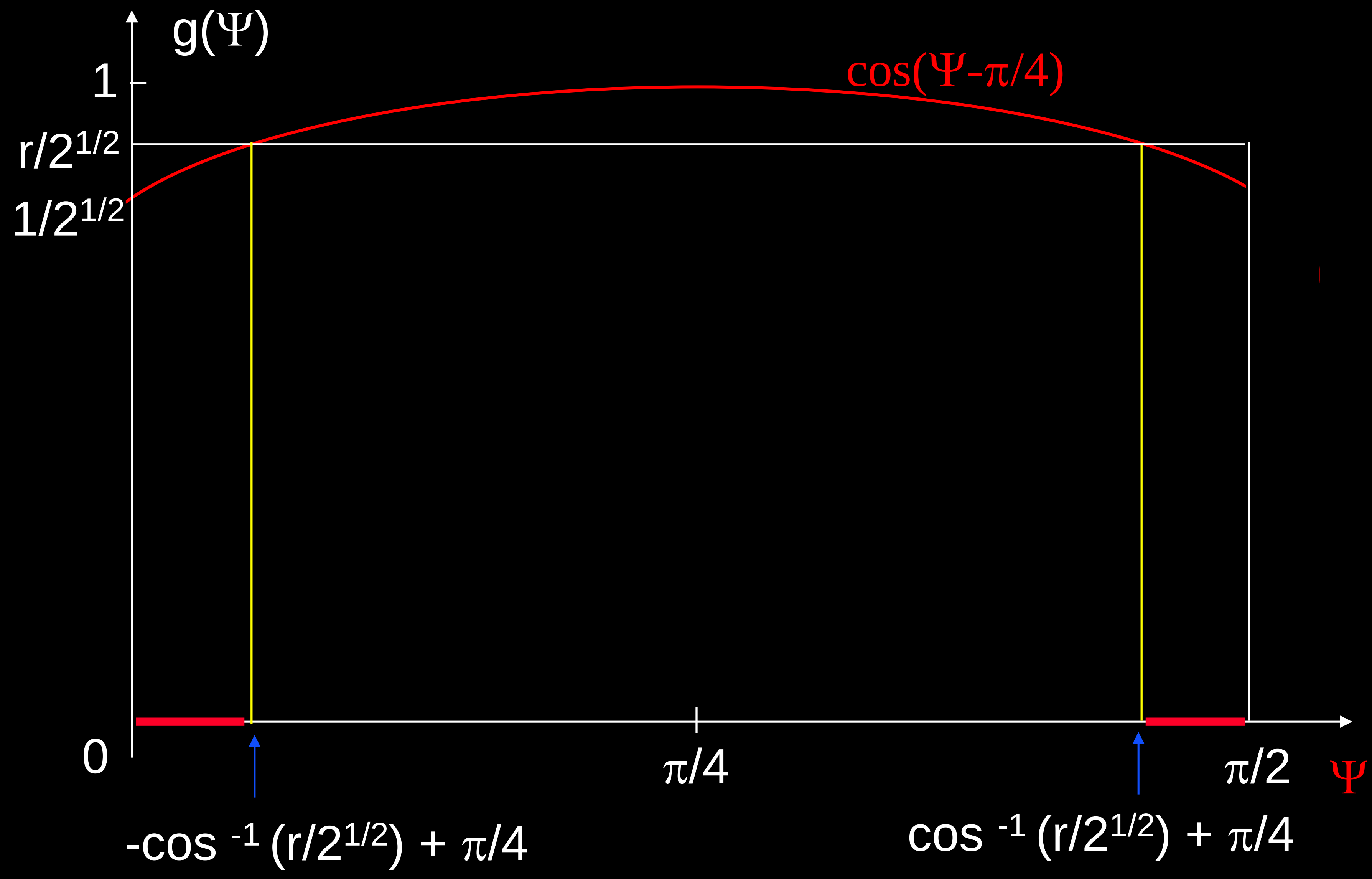


fΨ(ψ)
2/ π
0
π/2
ψ

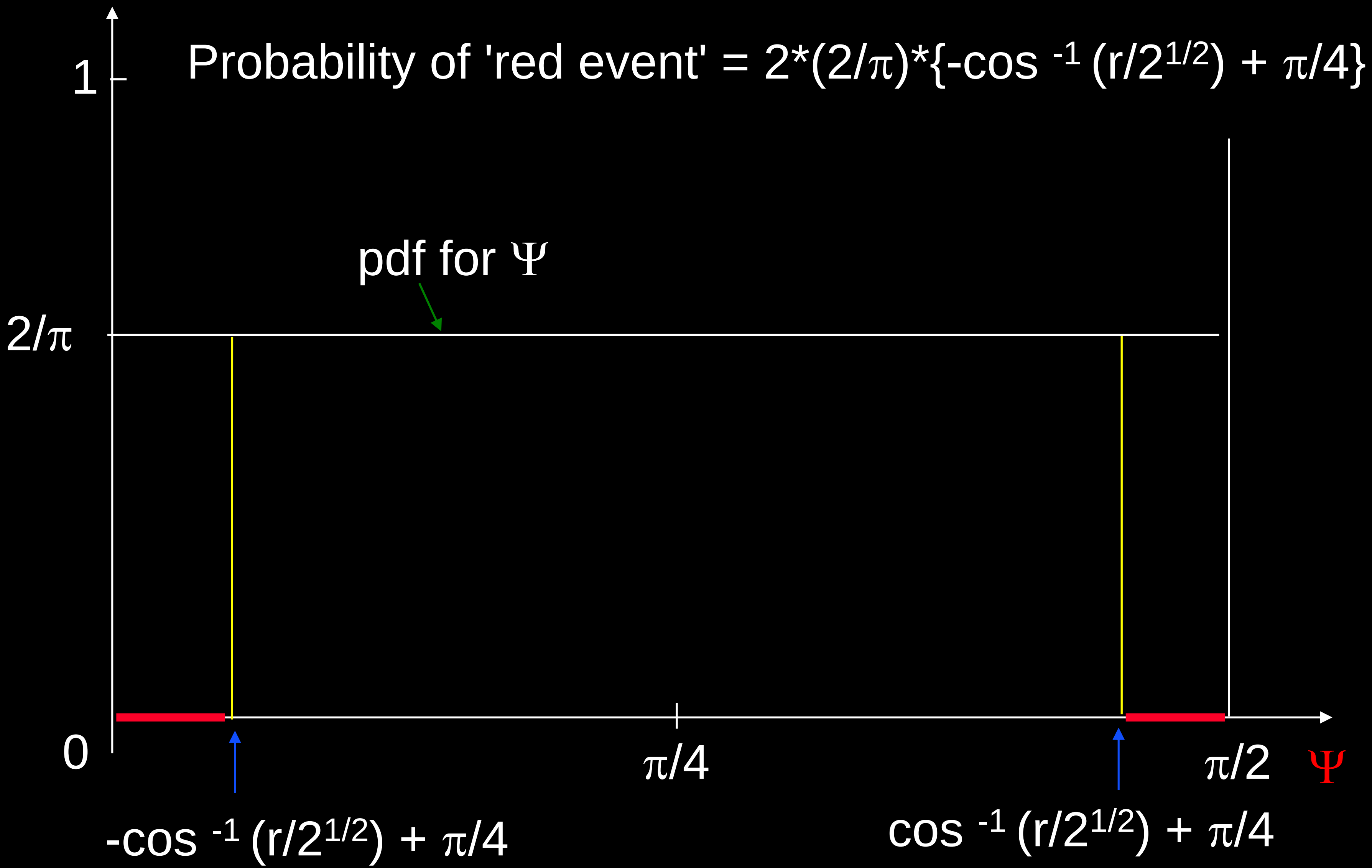
4. Find CDF

- $F_R(r) = P\{R < r\} = P\{2^{1/2} \cos(\Psi - \pi/4) < r\}$
- $F_R(r) = P\{R < r\} = P\{\cos(\Psi - \pi/4) < r/ 2^{1/2}\}$


g(Ψ)
1
r/2^1/2
1/2^1/2
cos(Ψ-π/4)
0
π/4
π/2
Ψ



g(Ψ)
1
r/2^{1/2}
1/2^{1/2}
cos(Ψ-π/4)
0
π/4
π/2
Ψ
-cos ^{-1} (r/2^{1/2}) + π/4
cos ^{-1} (r/2^{1/2}) + π/4

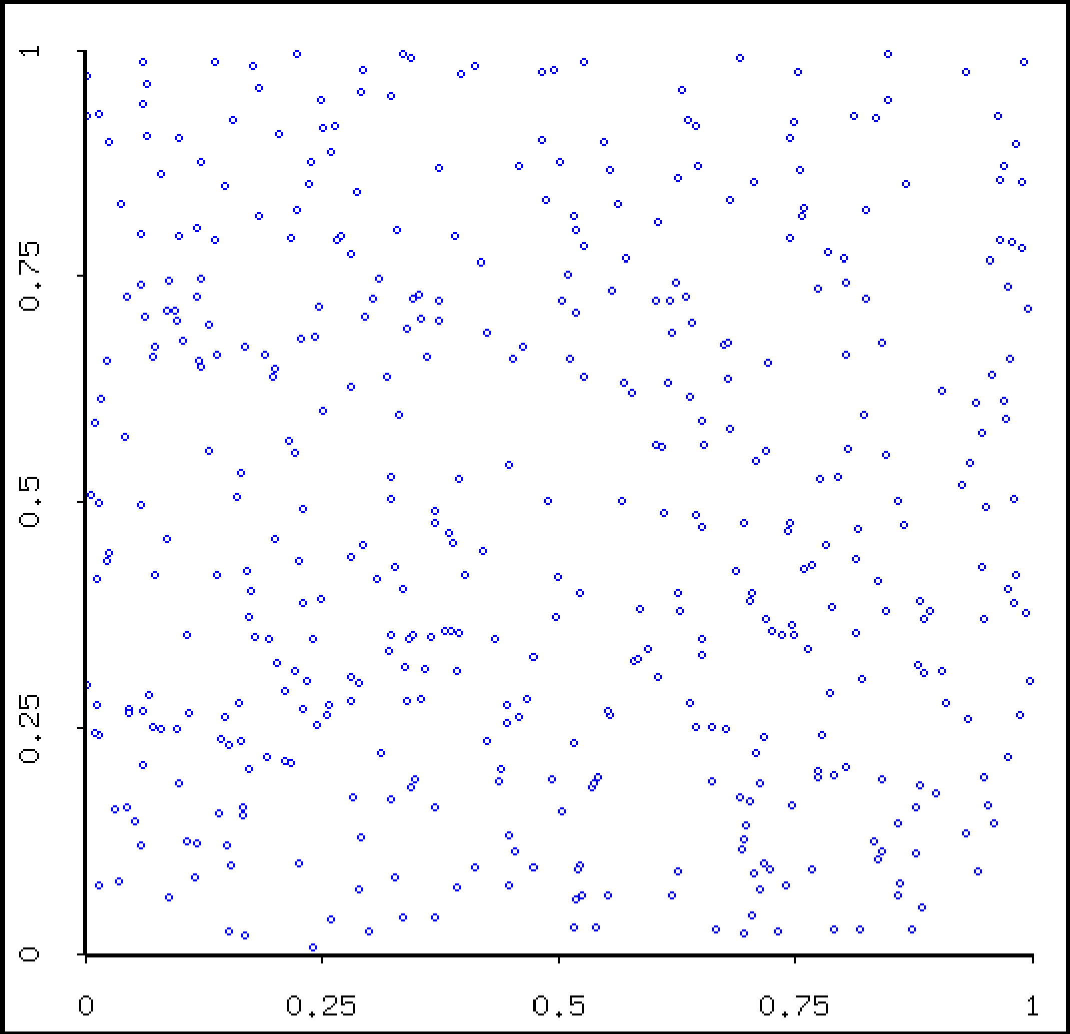


Probability of 'red event' = 2*(2/π)*{-cos -1 (r/2 1/2) + π/4}
1
pdf for Ψ
2/π
0
π/4
π/2
Ψ
-cos -1 (r/2 1/2) + π/4
cos -1 (r/2 1/2) + π/4

And finally...

- After all the computing is done, we find:
- $F_R(r) = 1 - (4/\pi)\cos^{-1}(r/2^{1/2}),\ \ 1 < r < 2^{1/2}$
- $f_R(r) = d[F_R(r)]/dr = (4/\pi)\{1/(2 - r^2)^{1/2}\}$
- Median R = 1.306
- $E[R] = 4/\pi = 1.273$
- $\sigma_R/E[R] = 0.098$, implies very robust

Spatial Poisson Processes



http://zappa.nku.edu/~longa/geomed/modules/ss1/lec/poisson.gif

Spatial Poisson Processes

- Entities distributed in space (Examples?)
- Follow postulates of the (time) Poisson process
  - *λdt = Probability of a Poisson event in dt*
  - *History not relevant*
  - *What happens in disjoint time intervals is independent, one from the other*
  - *The probability of a two or more Possion events in dt is second order in dt and can be ignored*

- Let's fill in the spatial analogue…..

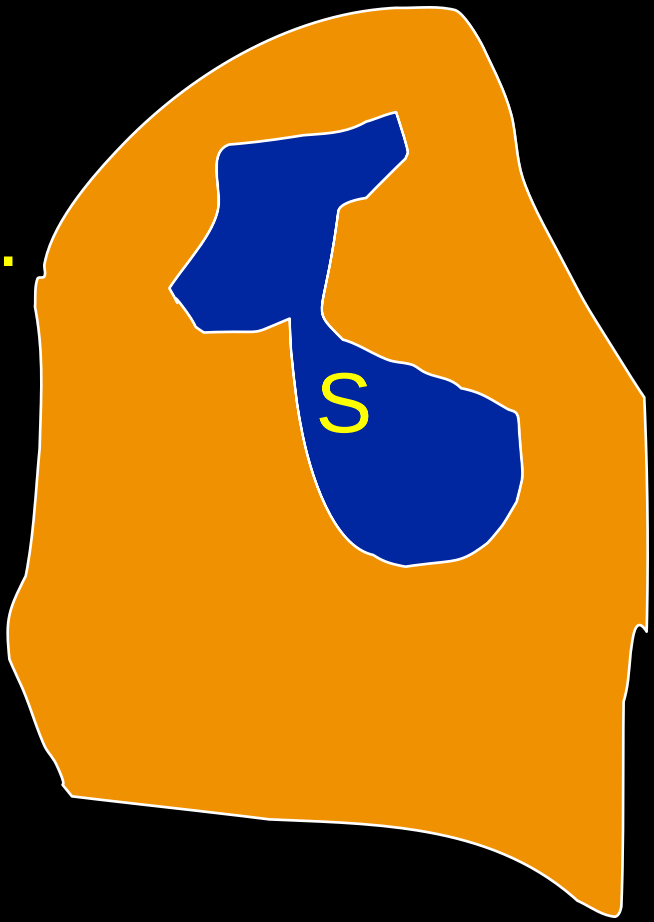
Set S has area $A(S)$.

Poisson intensity is γ

Poisson entities/(unit area).

$X(S)$ is a random variable

$X(S)$ = number of Poisson entities in S



$$P\{X(S)=k\}=\frac{(\gamma A(S))^k}{k!}e^{-\gamma A(S)},\ k=0,1,2,\ldots$$

Nearest Neighbors: Euclidean

Define D_2= distance from a random point to nearest Poisson entity

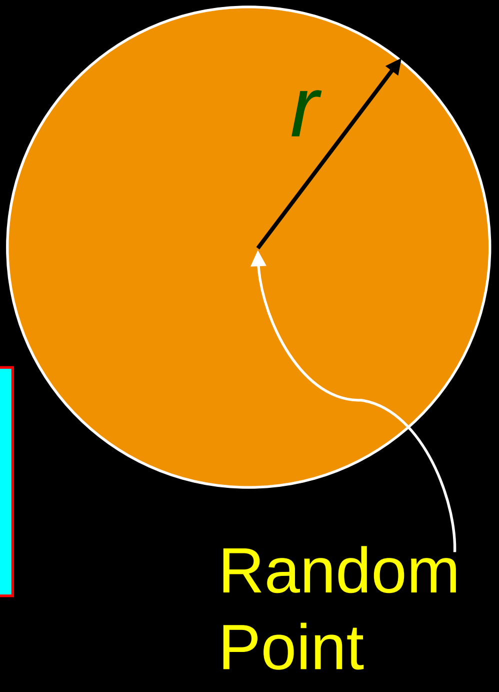
Want to derive $f_{D_2}(r)$.

Happiness:

$$F_{D_2}(r) \equiv P\{D_2 \le r\} = 1 - P\{D_2 > r\}$$

$$F_{D_2}(r) = 1 - \Pr ob\{\text{no Poisson entities in circle of radius } r\}$$

$$F_{D_2}(r) = 1 - e^{-\gamma \pi r^2} \quad r \ge 0$$



$$f_{D_2}(r) = \frac{d}{dr} F_{D_2}(r) = 2r\gamma\pi e^{-\gamma \pi r^2} \quad r \ge 0$$

Rayleigh pdf with parameter $\sqrt{2\gamma\pi}$

Nearest Neighbors: Euclidean

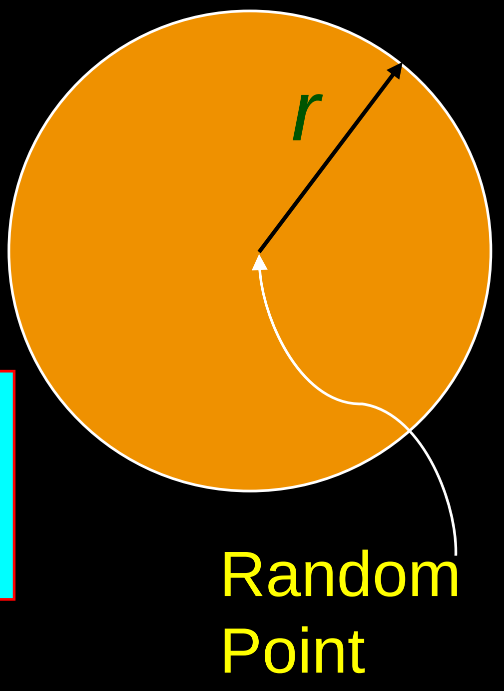
Define D_2 = distance from a random point to nearest Poisson entity

Want to derive $f_{D_2}(r)$.

$$E[D_2] = (1/2)\sqrt{\frac{1}{\gamma}} \quad \text{"Square Root Law"}$$

$$\sigma_{D_2}^2 = (2 - \pi/2)\frac{1}{2\pi\gamma}$$

$$f_{D_2}(r) = \frac{d}{dr}F_{D_2}(r) = 2r\gamma\pi e^{-\gamma\pi r^2} \quad r \geq 0$$

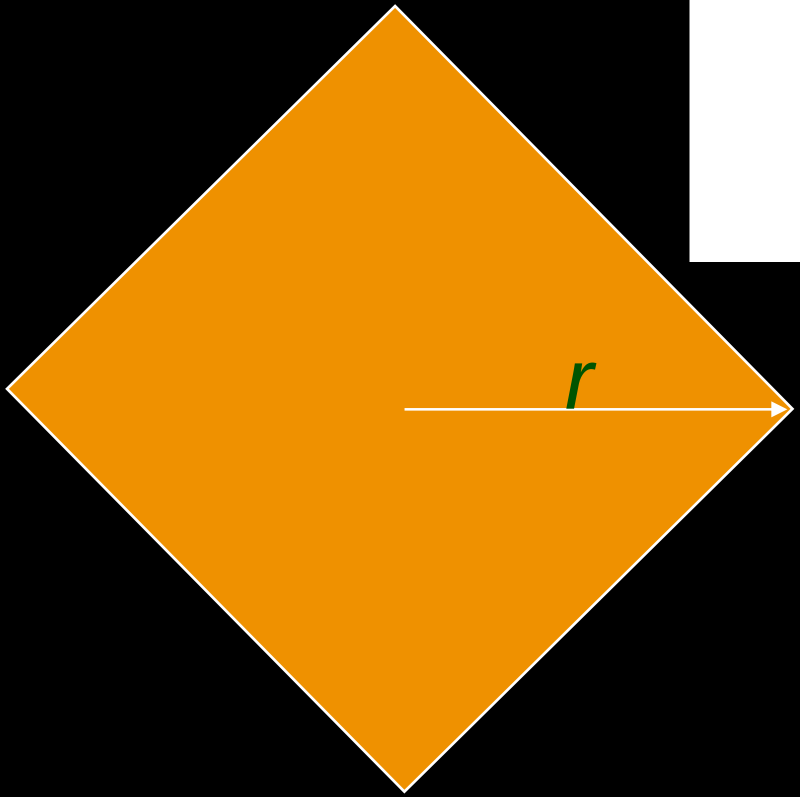


Rayleigh pdf with parameter $\sqrt{2\gamma\pi}$

Nearest Neighbor: Taxi Metric

$$F_{D_1}(r) \equiv P\{D_1 \leq r\}$$

$$F_{D_1}(r) = 1 - \Pr\{\text{no Poisson entities in diamond}\}$$



What Have We Learned?

- Within a spatial context, how to use the Four Steps to Happiness to derive joint distributions
- Within a spatial context, how to derive a difficult distribution involving geometry
- Spatial Poisson Processes, with nearest neighbor applications