# ESD.86
# The Weibull Distribution and Parameter Estimation

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# Weibull’s 1951 Paper

- “A Statistical Distribution Function of Wide Applicability”
- Journal of Applied Mechanics
- Key elements
  - A simple, but powerful mathematical idea
  - A method to reduce the idea to practice
  - A wide range of data
- Why study it in this course?

Weibull, W., 1951,“A Statistical Distribution Function of Wide Applicability,” *J. of Appl. Mech*.

# Signs of a Struggle

"The objection has been stated that this distribution function has no theoretical basis. But ... there are – with very few exceptions – the same objections against all other df, at least in so far as the theoretical basis has anything to do with the population in question. Furthermore, it is utterly hopeless to expect a theoretical basis for distribution functions of random variables such as ..."

Weibull, W., 1951,"A Statistical Distribution Function of Wide Applicability," *J. of Appl. Mech*.

# How was the Struggle Resolved?

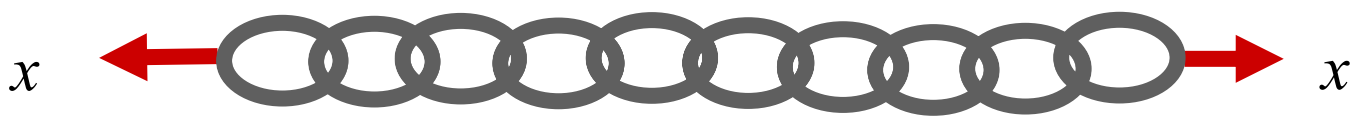
The reaction to his paper in the 1950s was negative, varying from skepticism to outright rejection... Weibull's claim that the data could select the distribution and fit the parameters seemed too good to be true. However, pioneers in the field like Dorian Shainin and Leonard Johnson applied and improved the technique. ... Today, Weibull analysis is the leading method in the world for fitting life data.

Abernathy, Robert, 2002, *The New Weibull Analysis Handbook*

# Weibull's Derivation

Call $P_n$ the probability that a chain will fail under a load of $x$

Let's define a cdf for *each* link meaning the link will fail at a load $X$ less than or equal to $x$ as $P(X \leq x) = F(x)$




If the chain does not fail, it's because all $n$ links did not fail

If the $n$ link strengths are probabilistically independent

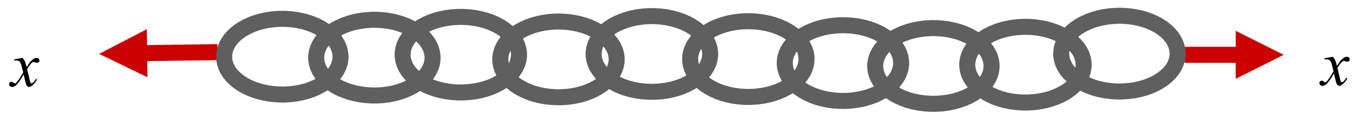
$$1 - P_n = (1 - P)^n$$

Weibull, W., 1951, "A Statistical Distribution Function of Wide Applicability," *J. of Appl. Mech*.

# Weibull's Derivation

A cdf can be transformed into the form $F(x) = 1 - e^{-\varphi(x)}$

This is convenient because $(1 - F(x))^n = (1 - P)^n = e^{-n\varphi(x)}$




The function $\varphi(x)$ must be positive, non-decreasing, and should vanish at some value $x_u$ which is often zero but not necessarily.

Among simplest functions satisfying the condition is $\varphi(x) = \frac{(x - x_u)^m}{x_o}$

So, a reasonable distribution to try is
$$F(x) = 1 - e^{-\frac{(x - x_u)^m}{x_o}}$$

# A Discussion Point

$$F(x) = 1 - e^{-\frac{(x - x_u)^m}{x_o}}$$

What is the probability of failure at a load of $x_u$?

Try to think of a situation where there is a physical or logical reason to suppose a non-zero value of $x_u$ exists.

# The Weibull Distribution

Weibull derived $F(x) = 1 - e^{-\frac{(x - x_u)^m}{x_o}}$

More common today to see

Location parameter

$$F(x) = 1 - e^{-\left(\frac{x-\theta}{\lambda}\right)^k}$$

Shape parameter

Scale parameter

OR

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}$$

other notations also used, be careful!

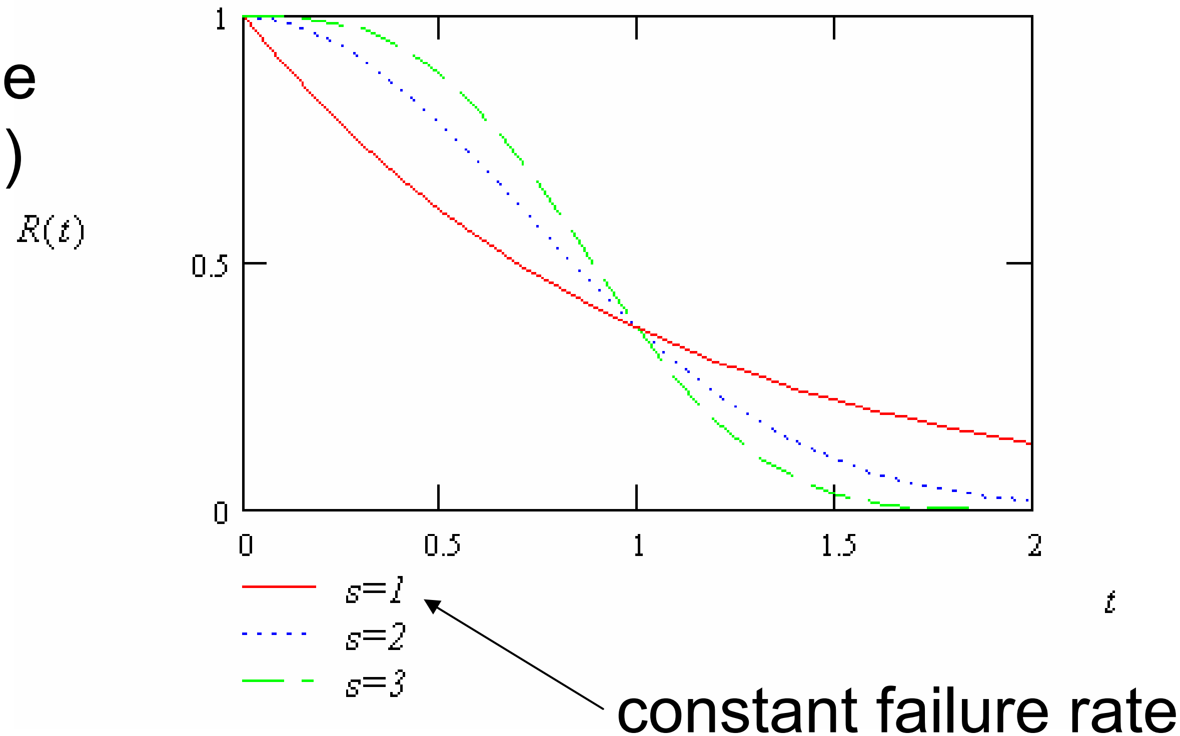
If location parameter=0, we call it the "two parameter" Weibull distribution

Weibull reported for Bofors steel $m$=2.93. What is $k$ or $\alpha$?

# Influence of the Shape Parameter

In reliability, the following "reliability function" is commonly defined (the complement of failure)

$$R(t) = e^{-\left(\frac{t - t_o}{\eta}\right)^s}$$



# Useful Facts about the Two Parameter Weibull Distribution

cdf $F(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^k}$

Support

$x \in [0;+\infty)$

pdf $f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}$

Mean $\lambda\Gamma\left(1+\frac{1}{k}\right)$

Variance $\lambda^2\Gamma\left(1+\frac{2}{k}\right) - \mu^2$

Median $\lambda \ln(2)^{1/k}$

Skew $\dfrac{\lambda^3\Gamma\left(1+\frac{3}{k}\right) - 3\mu\sigma^2 - \mu^3}{\sigma^3}$

How is this slide different for the three parameter distribution?

# The Procedure Weibull used for Parameter Estimation

If we assume $F(x) = 1 - e^{-\frac{(x-x_u)^m}{x_o}}$

It follows that $$\log\left[\log\left(\frac{1}{1-F(x)}\right)\right] = m\log(x - x_u) + \log\left(\frac{1}{x_0}\right)$$

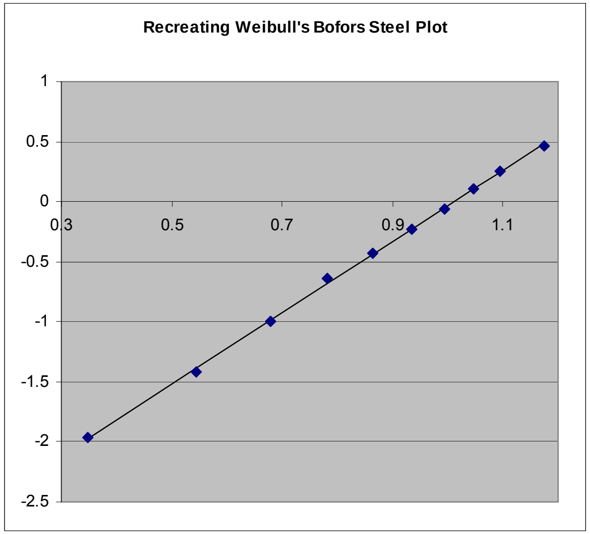
Which is in the form of a linear realtionship, so Weibull:

1. Starts with a list of values $x$ for strength, size, life ...
2. Assigns observed probabilities $P \approx F(x)$ to the values
3. Transforms the $P$ and $x$ values as indicated above
4. Fits a straight line to the data

Note: How do we estimate this value?

| $x$ [1.275 $kg/mm^3$] | $n$ | $P$ | $\log\left[\log\left(\frac{1}{1-P}\right)\right]$ | $\log(x\text{-}x_u)$ |
|---|---|---|---|---|
| 32 | 10 | | | |
| 33 | 36 | | | |
| 34 | 84 | | | |
| 35 | 150 | | | |
| 36 | 224 | | | |
| 37 | 291 | | | |
| 38 | 340 | | | |
| 39 | 369 | | | |
| 40 | 383 | | | |
| 42 | 389 | | | |

How do we estimate this probability?
What are the potential bear-traps?



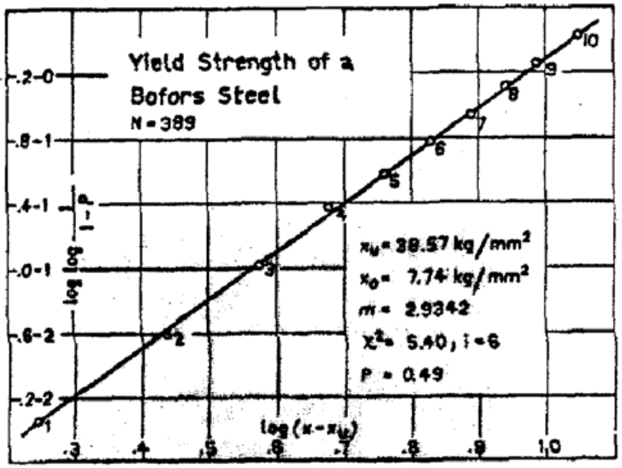
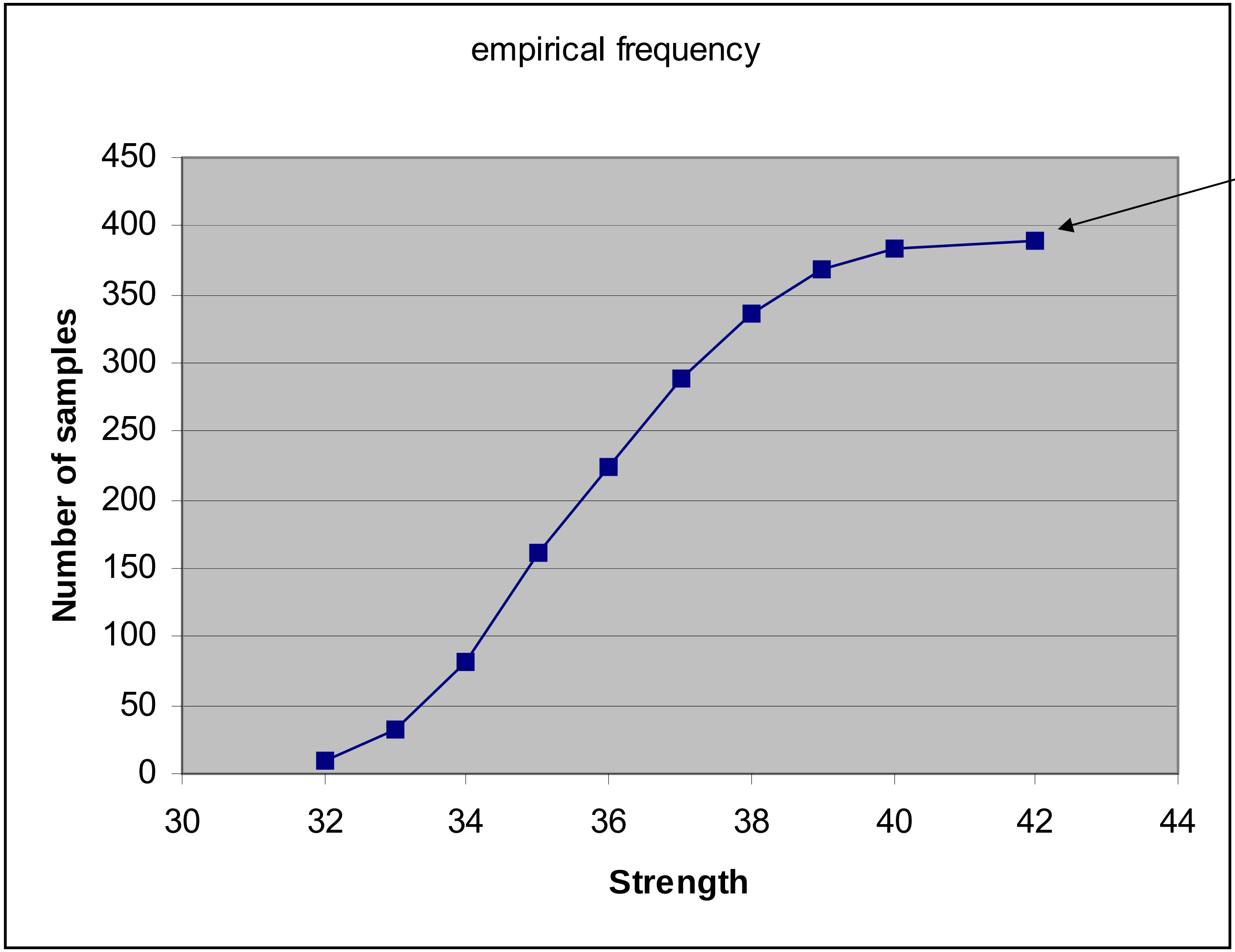




FIG. 1 YIELD STRENGTH OF A BOFORS STEEL



Weibull, W., 1951,"A Statistical Distribution Function of Wide Applicability," *J. of Appl. Mech*.

# Plot of Weibull's Data




What $P$ should we list for $x$ value 42?

The convention for probabilty plotting is $(i\text{-}1/2)/n$

# Test for "Goodness of Fit" as Conducted in Weibull's Paper

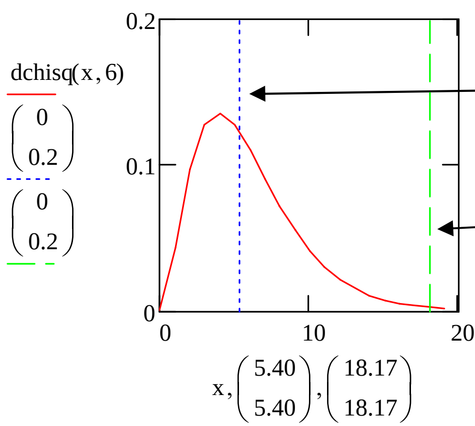
- Calculates the degrees of freedom

  10 (bins) -1 – 3 (parameters of the df) = 6

- Calculates the statistic

$$\chi^2 = \sum \frac{(observed - estimated)^2}{estimated}$$

- States the *P*-value
- Comparison to alternative




*P=0.49* for Weibull

*P=0.008* for Normal

TABLE 1 YIELD STRENGTH OF A BOFORS STEEL

($x$ = yield strength in 1.275 kg/mm²)

| | $x$ | Expected values $n$ | Observed values $n$ | Normal distribution $n$ |
|---|---|---|---|---|
| 1 | 32 | 10 | 10 | 8 |
| 2 | 33 | 36 | 33 | 28 |
| 3 | 34 | 84 | 81 | 71 |
| 4 | 35 | 150 | 161 | 141 |
| 5 | 36 | 224 | 224 | 225 |
| 6 | 37 | 291 | 289 | 301 |
| 7 | 38 | 340 | 336 | 351 |
| 8 | 39 | 369 | 369 | 376 |
| 9 | 40 | 383 | 383 | 386 |
| 10 | 42 | 389 | 389 | 388 |

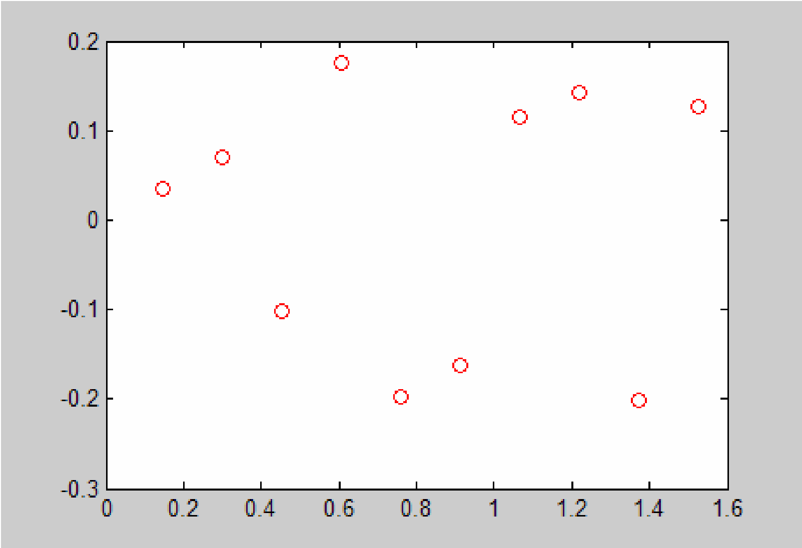


Note: Table is cumulative, $\chi^2$test requires <u>frequency in bin</u>

# Another Good Check on Fit

- Plot the residuals
  - Check for patterns
  - Check for uniform variance

```
e=y-y_hat;
plot(y_hat, e, 'or')
```



# Simple Distributions

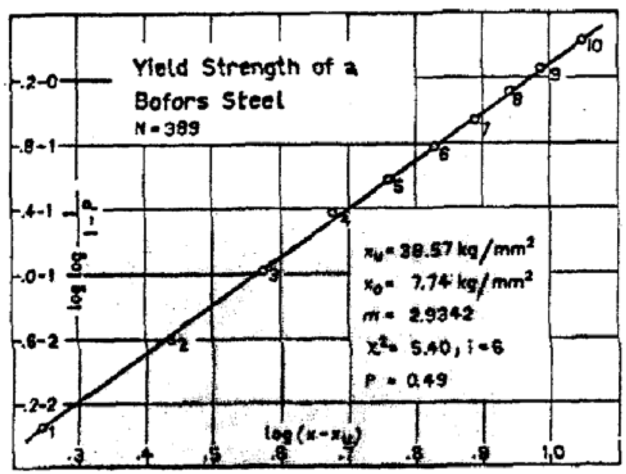



Fig. 1 Yield Strength of a Bofors Steel



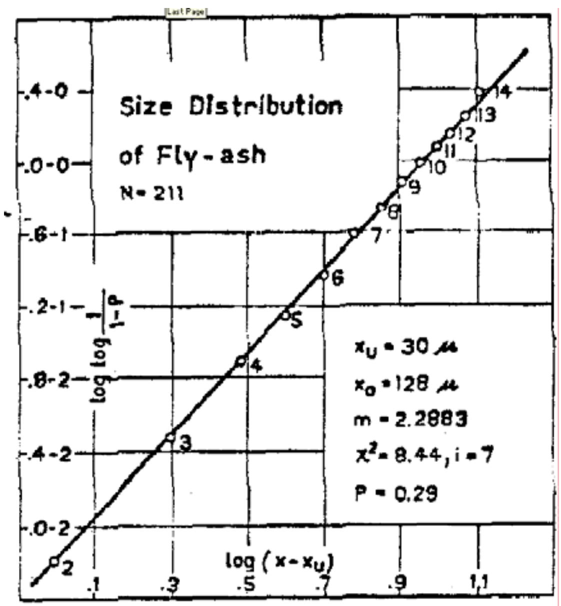



Fig. 2 Size Distribution of Fly Ash



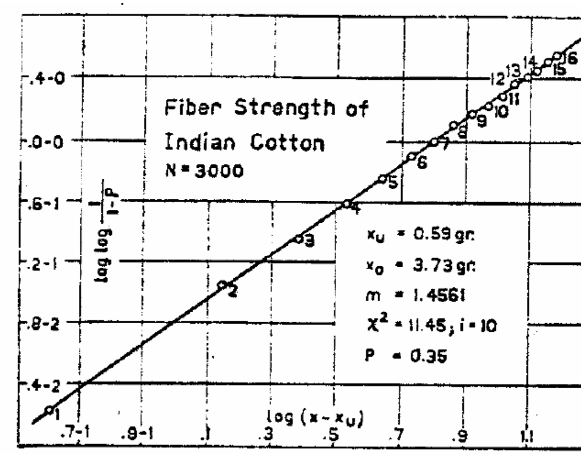



Fig. 3 Fiber Strength of Indian Cotton



Weibull, W., 1951,"A Statistical Distribution Function of Wide Applicability," *J. of Appl. Mech*.

# Maximum Likelihood Estimates

- Choose the estimate $\hat{\theta}$ of the parameter $\theta$ to maximize the likelihood function

$$L(\theta) = f(X_1, X_2, \ldots, X_n; \theta)$$

$n$ observed values

parameter value(s) to be estimated

- Or, if more convenient, maximize $\log(\mathrm{L}(\theta))$

Note: The estimated parameter value is not guaranteed to be the most likely one. It's <u>the data</u> that is made most likely by the parameter estimate.

# Maximum Likelihood Estimate of the Weibull Distribution

- Write the likelihood function

$$L(k,\lambda) = f(X_1, X_2, \ldots, X_n; k, \lambda) = \prod_{i=1}^{n} \frac{k}{\lambda}\left(\frac{X_i}{\lambda}\right)^{k-1} e^{-\left(\frac{X_i}{\lambda}\right)^{k}}$$

$$\frac{\partial}{\partial \lambda}\ln\big(L(k,\lambda)\big) = 0$$ leads to $$\lambda = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\,k}\right)^{1/k}$$

$$\frac{\partial}{\partial k}\ln\big(L(k,\lambda)\big) = 0$$ leads to $$k = \left(\frac{\sum X_i^{\,k}\ln(X_i)}{\sum X_i^{\,k}} - \frac{\sum \ln(X_i)}{n}\right)^{-1}$$

Note: But there is no closed form solution in general and numerical methods must be used.

# Some Terms Related to Estimation

- Consistent – for any $c$ $\lim_{n\to\infty} P\left(\left|\widehat{\theta}-\theta\right| \geq c\right) = 0$ MLEs are
- Unbiased – $E(\widehat{\theta}) = \theta$ MLEs are not always
- Minimum variance

$$\operatorname{var}(\widehat{\theta}) = \frac{1}{nE\left[\left(\frac{\partial \ln f(X)}{\partial \theta}\right)^2\right]}$$

MLEs are pretty close

# Concept Question

```
number_of_trials=10;
number_of_samples=10;
for t=1:number_of_trials
    data = wblrnd(2.0,0.8,number_of_samples,1);
    [paramhat, paramci] = wblfit(data);
    shape(t)=paramhat(1); scale(t)=paramhat(2);
end
mean(shape)
mean(scale)
```

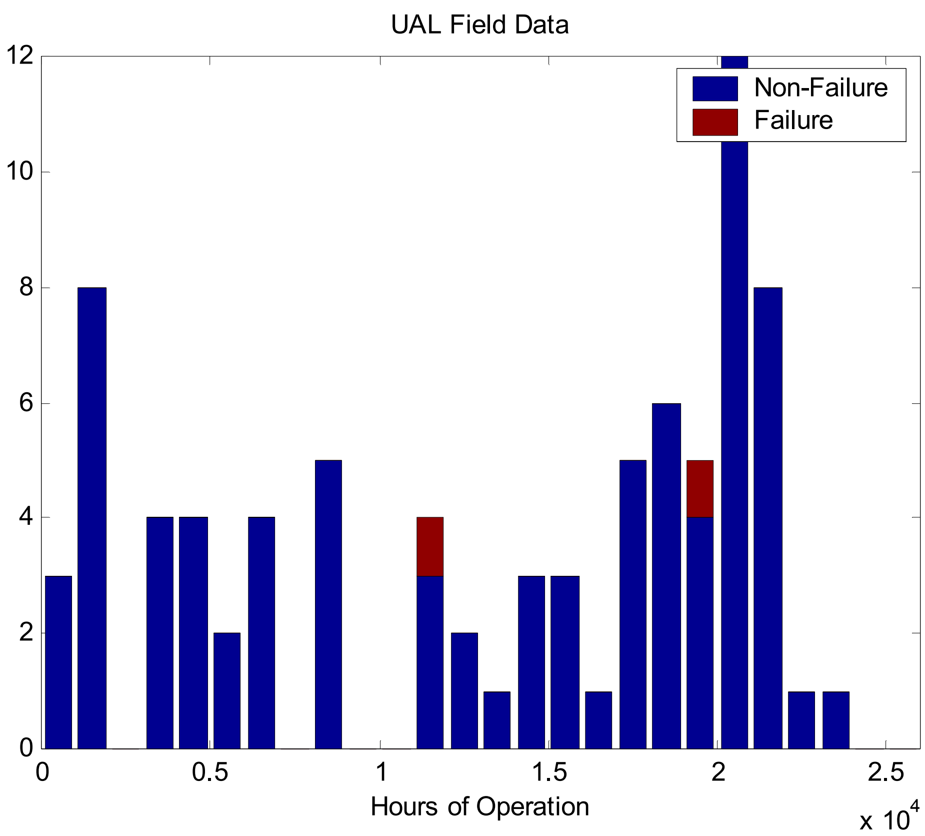
To make the mean(paramhat) equal to the parameters:

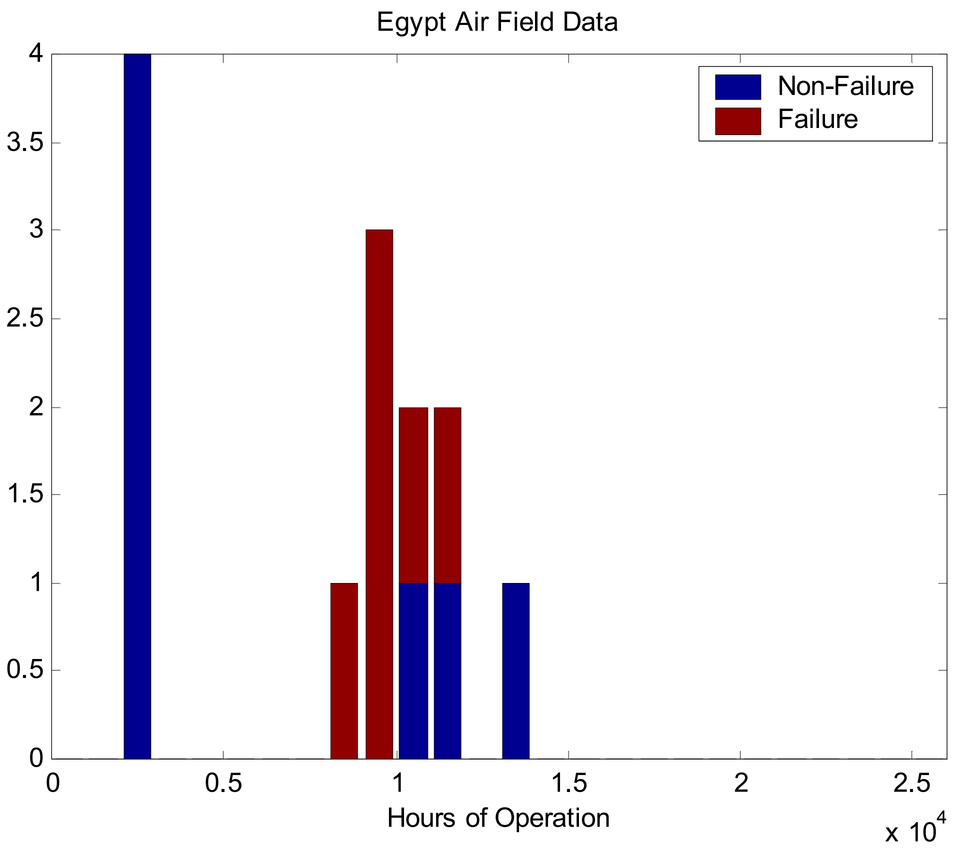
1) raise number_of_trials
2) raise number of samples
3) both must be raised
4) will not converge exactly in the limit even if both $\rightarrow\infty$

# Point and Interval Estimates

- Up to now, we have discussed point estimates only – a single real value for a parameter
- These are fine, but sometimes one would like to communicate information about degree of confidence
- For this, interval estimates are helpful
- e.g., ±95% confidence intervals on paramters

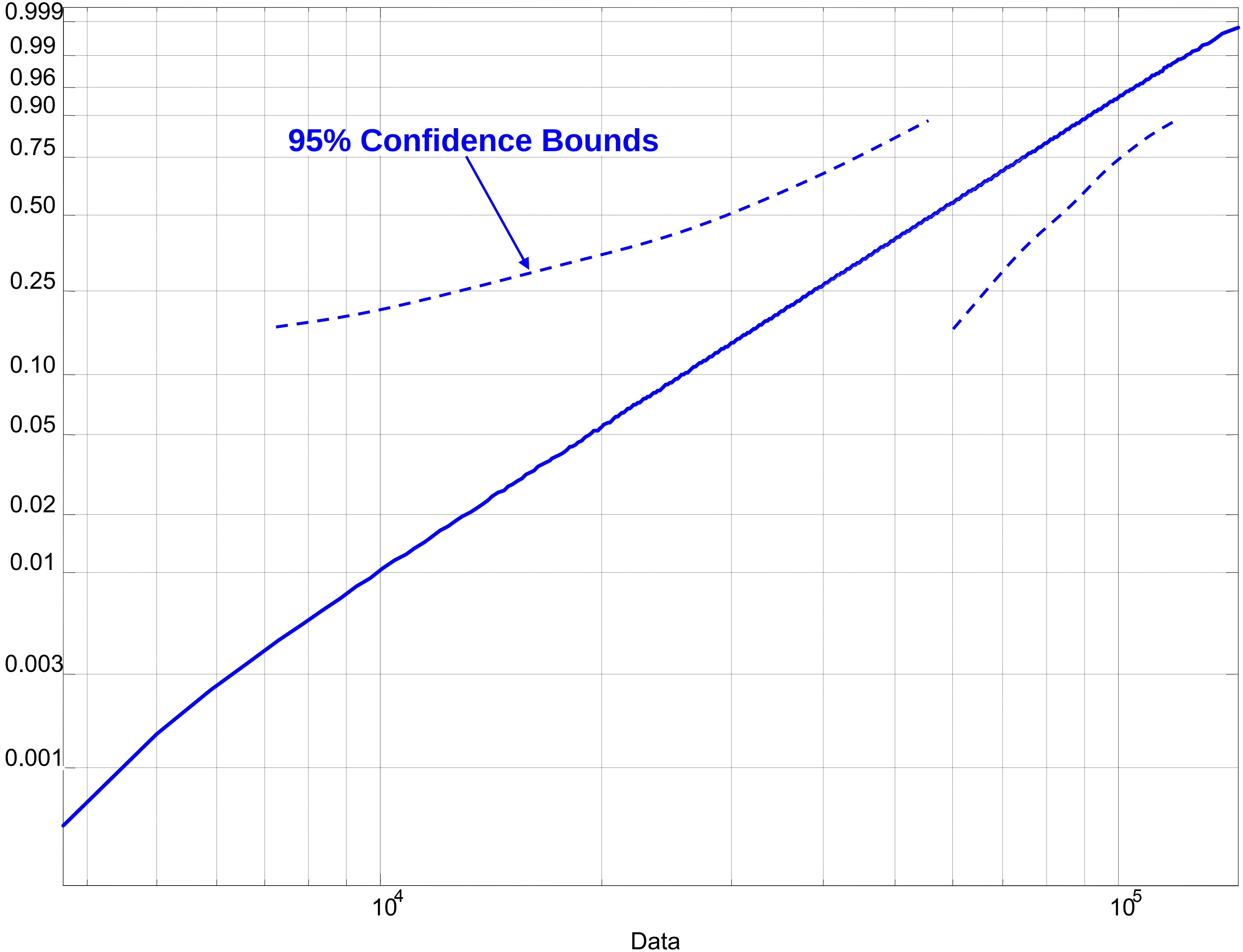
# Field Data on Engine Life



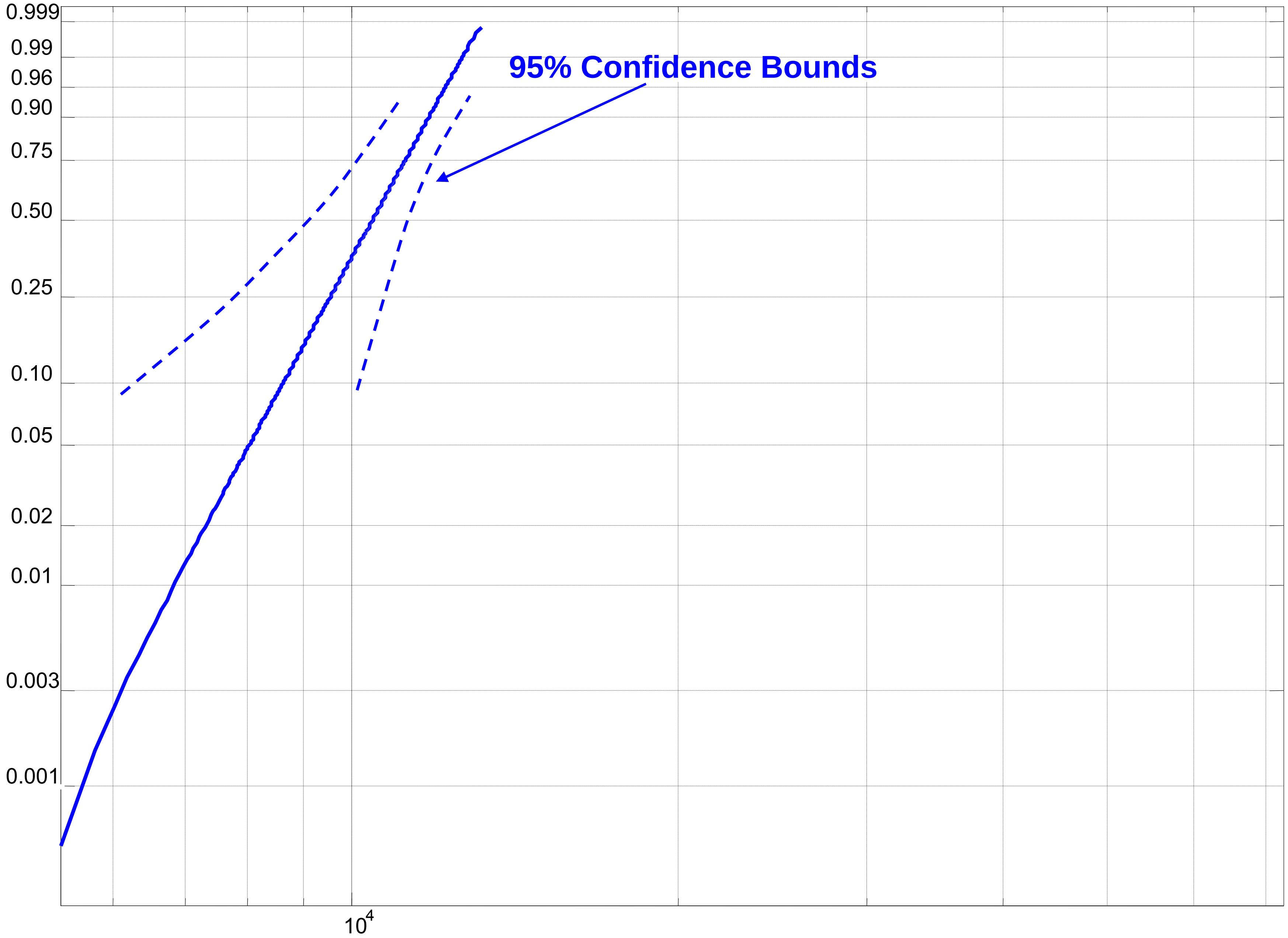




NOTE: What should we do about the non-failed items? What do we lose if we censor the data?

# UAL Field Failure Weibull Plot



# Egypt Air Field Failure Weibull Plot



# Complex Distributions

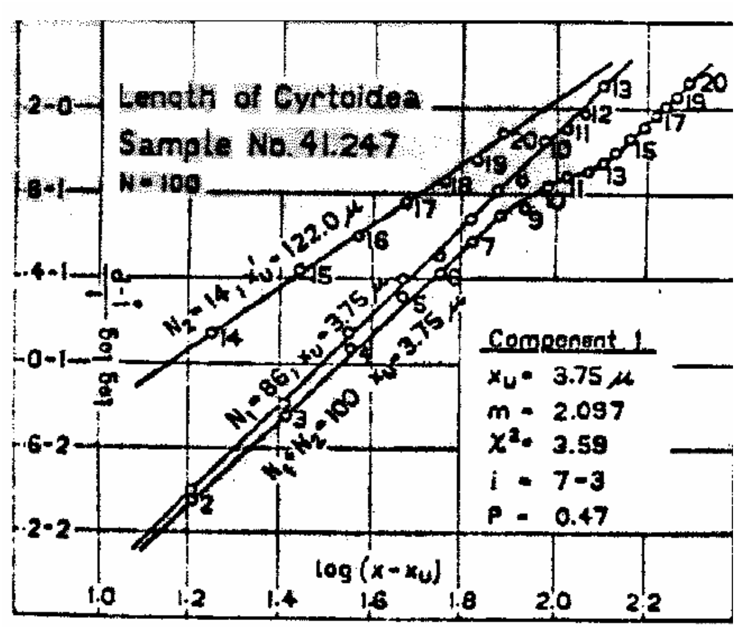



FIG. 4 LENGTH OF CYRTOIDEAE

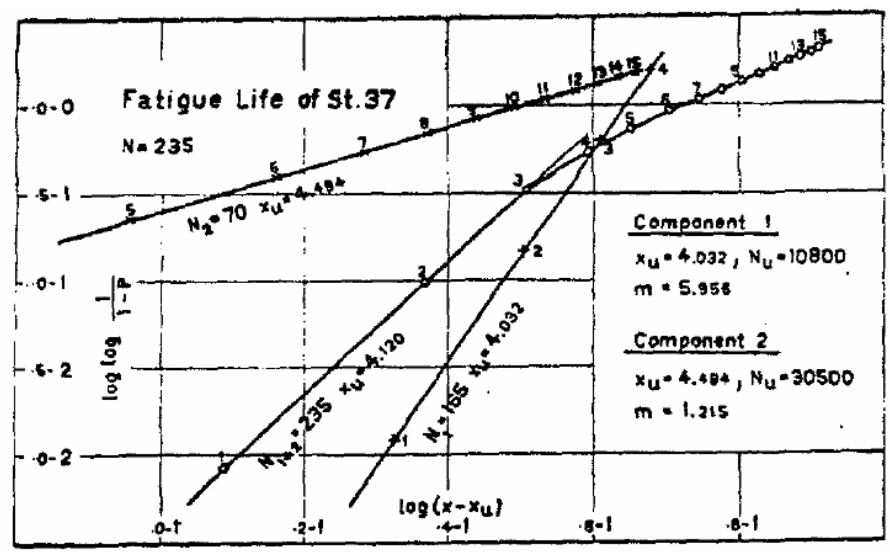



FIG. 6 FATIGUE LIFE OF ST-37 STEEL





Weibull, W., 1951,“A Statistical Distribution Function of Wide Applicability,” *J. of Appl. Mech*.

# Looking for Further Evidence of Two Populations

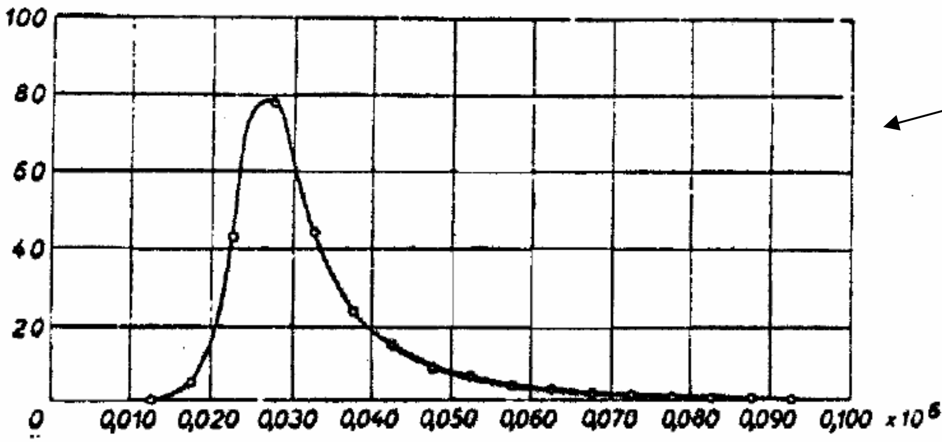



FIG. 5 FREQUENCY CURVE OF FATIGUE LIFE OF ST-37
(Number of specimens versus number of stress cycles.)



No evidence of bi-modality in fatigue data

Clear evidence of bi-modality in strength data

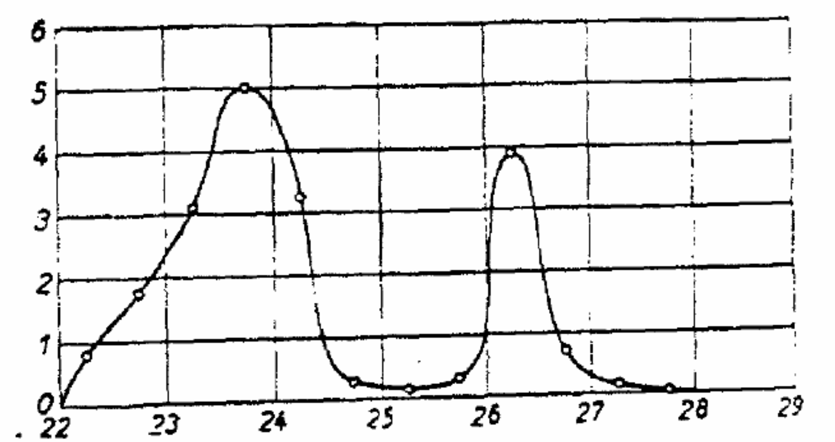



FIG. 7 FREQUENCY CURVE OF YIELD STRENGTH OF ST-37 STE
(Number of specimens versus yield strength in kg/mm².)

# Reliability Terminology

- Reliability function *R*(*t*) -- The probability that a product will continue to meet its specifications over a time interval
- Mean Time to Failure *MTTF* -- The average time *T* before a unit fails

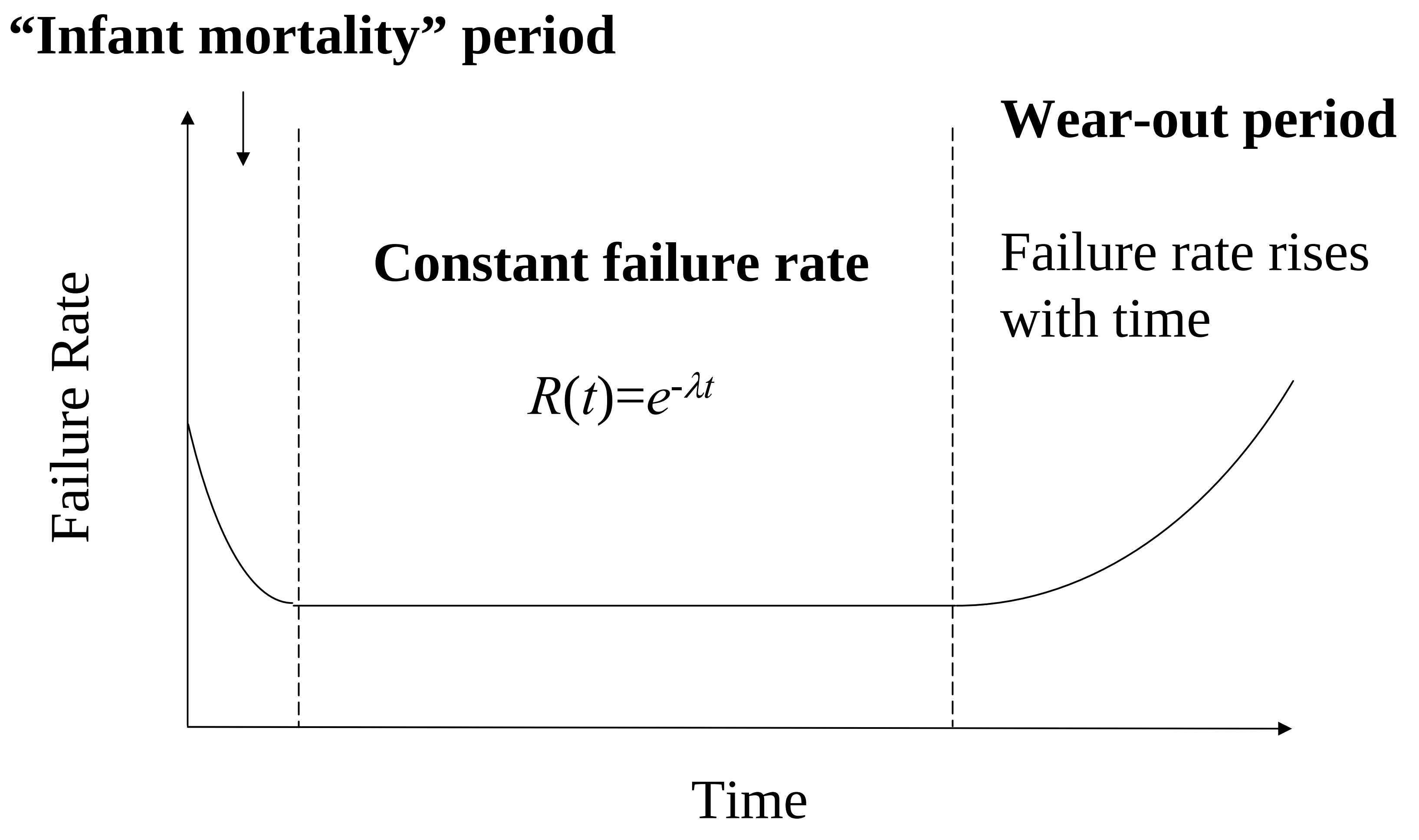
$$MTTF = \int_0^\infty R(t)dt$$

- Instantaneous failure rate $\lambda(t)$

$$\lambda(t) = \Pr(\text{System survives to } t + dt \,|\, \text{System survives to } t)$$

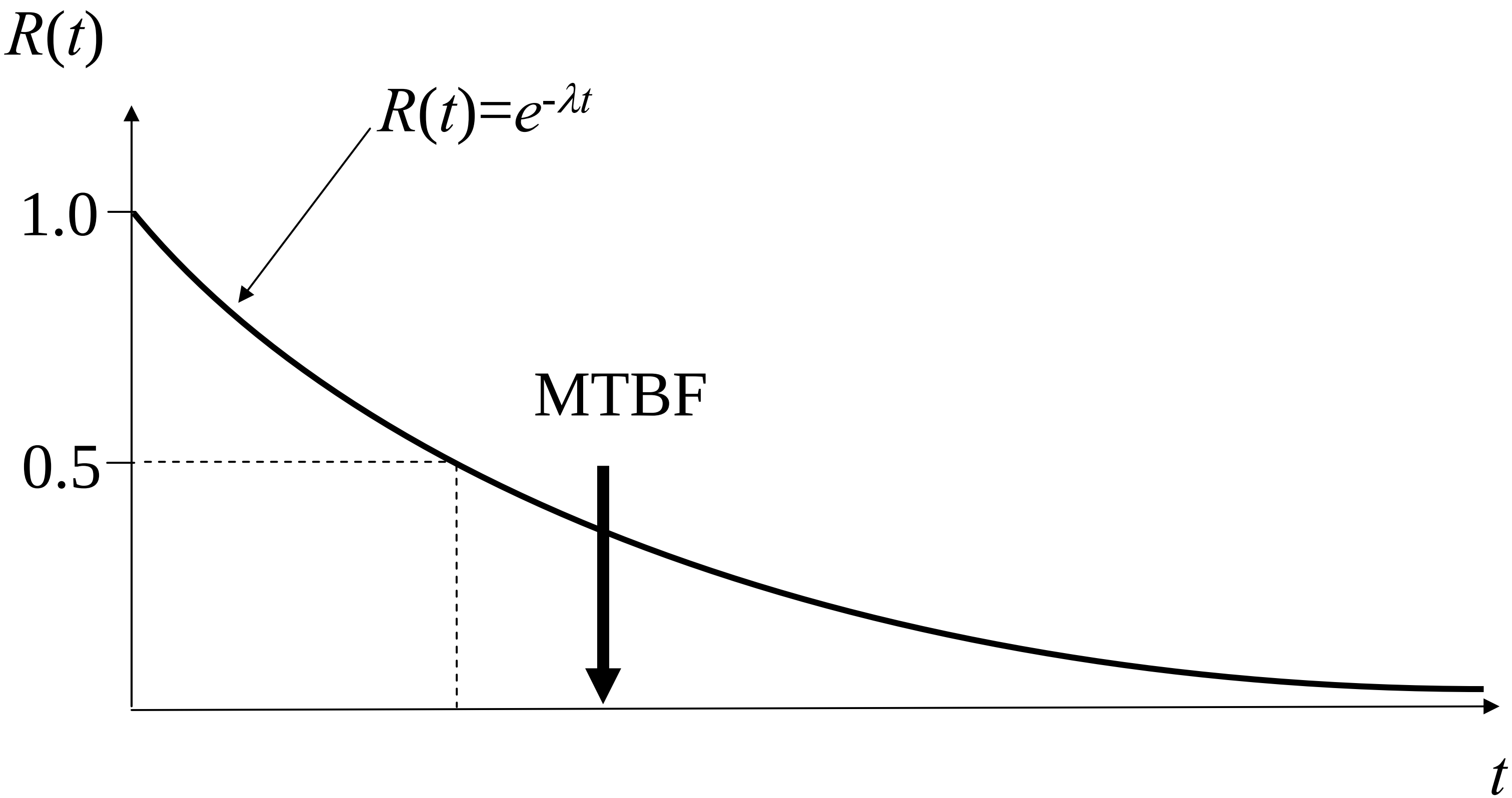
$$R(t) = e^{-\int_0^t \lambda(\xi)d\xi}$$

# The "Bathtub" Curve

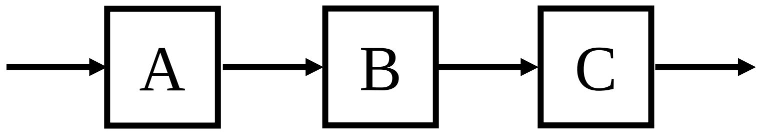


# Constant Failure Rates

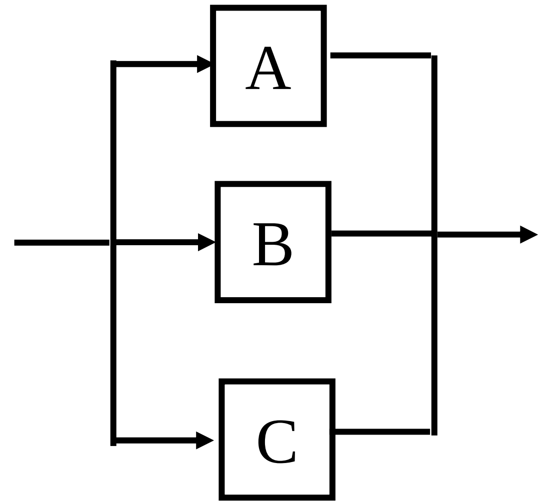
"When the system operating time is the MTBF, the reliability is 37%"
- Blanchard and Fabrycky



# Series and Parallel Networks




$$R = R_A R_B R_C$$

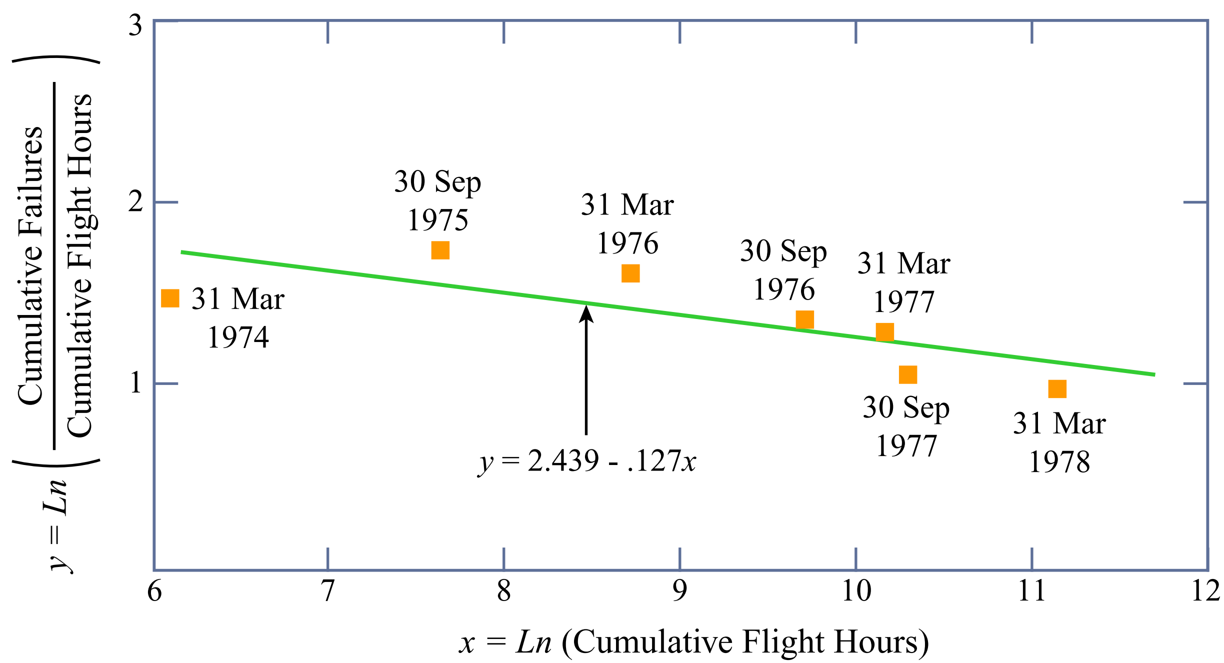



$$(1 - R) = (1 - R_A)(1 - R_B)(1 - R_C)$$

But ONLY if statistically independent!

# Reliability Growth (Duane Model)

- As newly designed equipment is refined, it becomes more reliable
- J. T Duane [1964] published regressions for aerospace items




Reliability improvement of the F-15A, an example of Duane model growth.

Figure by MIT OCW.

See Ushakov,1994, *Handbook of Reliability Engineering*

# Problem Set #5

1. Parameter estimation

    Make a probability plot

    Make an estimate by regression

    Make an MLE estimate

    Estimate yet another way

    Comment on "goodness of fit"

2. Hypothesis testing

    Find a journal paper uing the "null ritual"

    Suggest improvements (validity, insight, communication)

# Next Steps

- Between now and Weds
  - Read Gigerenzer "Mindless Satatistics"
- Wednesday 10:30-noon
  - Session on Hypothesis testing
- Friday
  - Recitation to support PS#5
- Monday
  - PS#5 due
  - Session on XXX

# The Wonderful One-Hoss Shay

*by Oliver Wendell Holmes*

- HAVE you heard of the wonderful one-hoss-shay, that was built in such a logical way it ran a hundred years to a day…
- Now in building of chaises, I tell you what, there is always somewhere a weakest spot,-- in hub, tire, felloe, in spring or thill, in panel, or crossbar, or floor, or sill…

But the Deacon swore .. he would build one shay to beat the taown 'n' the keounty 'n' all the kentry raoun'; It should be so built that it couldn' break daown! --"Fur," said the Deacon, "t 's mighty plain thut the weakes' place mus' stan' the strain; 'n' the way t' fix it, uz I maintain, is only jest t' make that place uz strong uz the rest"…

So the Deacon inquired of the village folk where he could find the strongest oak, That could n't be split nor bent nor broke,--

Eighteen hundred and twenty came;-- Running as usual; much the same. Thirty and forty at last arrive, And then come fifty, and fifty-five…

There are traces of age in the one-hoss-shay– A general flavor of mild decay, But nothing local, as one may say. There couldn't be,--for the Deacon's art had made it so like in every part that there wasn't a chance for one to start…

And yet, as a whole, it is past a doubt in another hour it will be worn out!

- First a shiver, and then a thrill, Then something decidedly like a spill,-- …What do you think the parson found, when he got up and stared around? The poor old chaise in a heap or mound, as if it had been to the mill and ground! You see, of course, if you're not a dunce, how it went to pieces all at once,-- all at once, and nothing first,-- just as bubbles do when they burst.
- End of the wonderful one-hoss-shay. Logic is logic. That's all I say.