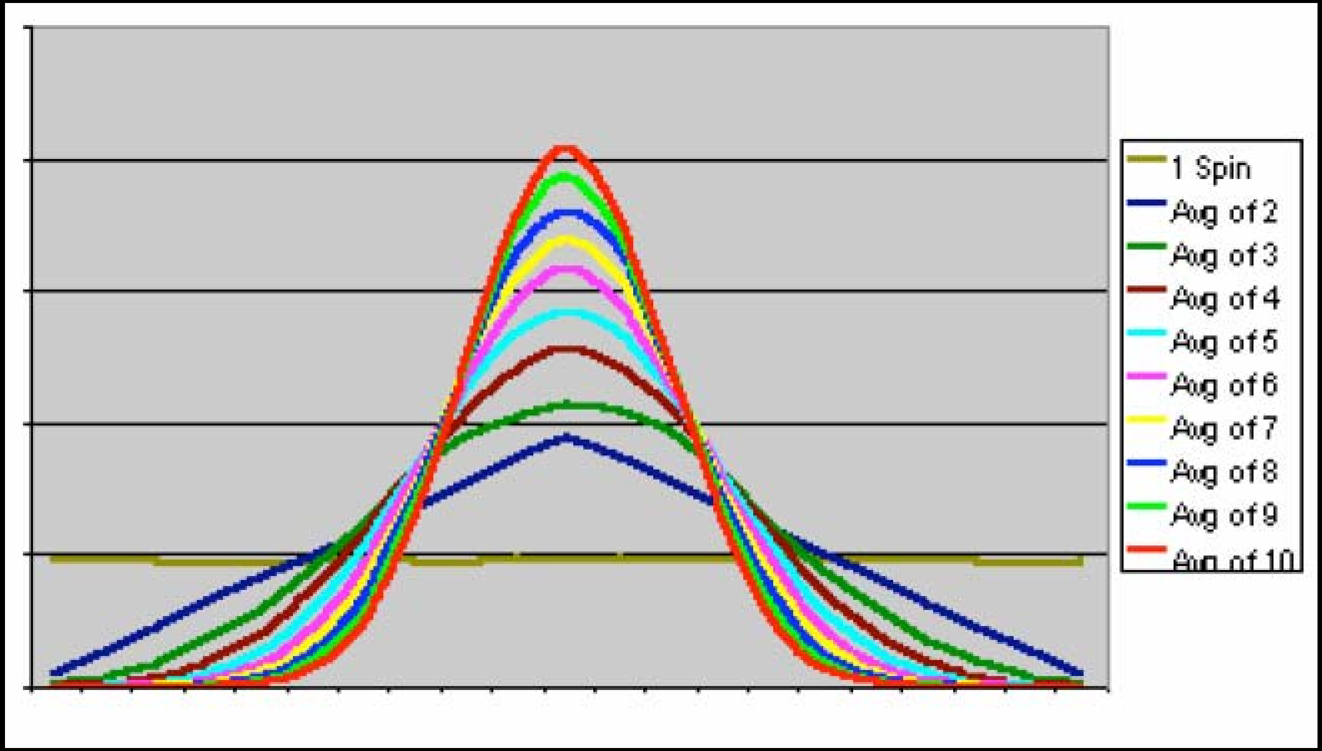
ESD.86. Queueing & Transitions

Sampling from Distributions, Gauss



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http://simulationtutorials.com/images/CLT.gif

Outline

- One More Time: Markov Birth and Death Queueing Systems
- Central Limit Theorem
- Monte Carlo Sampling from Distributions
- ‘Q&A’

Buy one, get the other 3 for free!

$$W = \frac{1}{\mu} + W_q$$

$$L = L_q + L_{SF} = L_q + \frac{\lambda}{\mu}$$

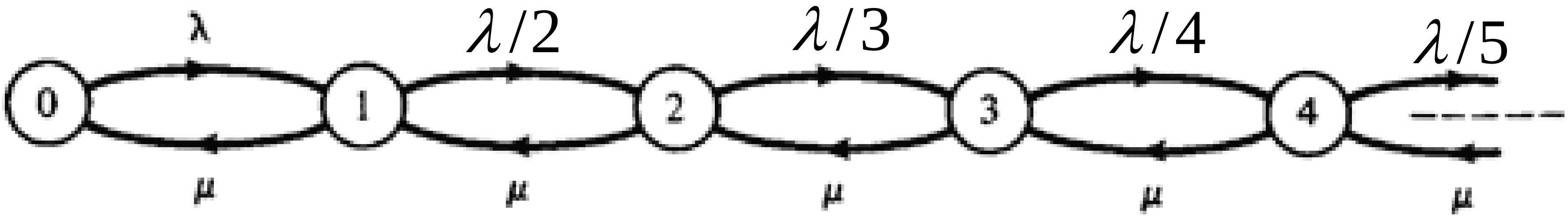
$$L = \lambda W$$

Optional Exercise:

Is it "better" to enter a single server queue with service rate μ or a 2-server queue each with rate $\mu/2$?

Can someone draw one or both of the state-rate-transition diagrams?
Then what do you do?

Final Example: Single Server, Discouraged Arrivals



State-Rate-Transition Diagram, Discouraged Arrivals

$$P_k = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k P_0$$

$$P_0 = \left[1 + \left(\frac{\lambda}{\mu}\right) + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 + \ldots + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k + \ldots\right]^{-1}$$

$$P_0 = (e^{\lambda/\mu})^{-1} = e^{-\lambda/\mu} > 0$$

$$P_0 = (e^{\lambda/\mu})^{-1} = e^{-\lambda/\mu} > 0$$

$$\rho = \text{utilization factor} = 1 - P_0 = 1 - e^{-\lambda/\mu} < 1.$$

$$P_k = \frac{(\lambda/\mu)^k}{k!} e^{-\lambda/\mu}, \quad k = 0,1,2,\ldots \text{ Poisson Distribution!}$$

$$L = \text{time-average number in system} = \lambda/\mu \quad \text{How?}$$

$$L = \lambda_A W \qquad \text{Little's Law, where}$$

$$\lambda_A \equiv \text{average rate of accepted arrivals into system}$$

Apply Little's Law to Service Facility

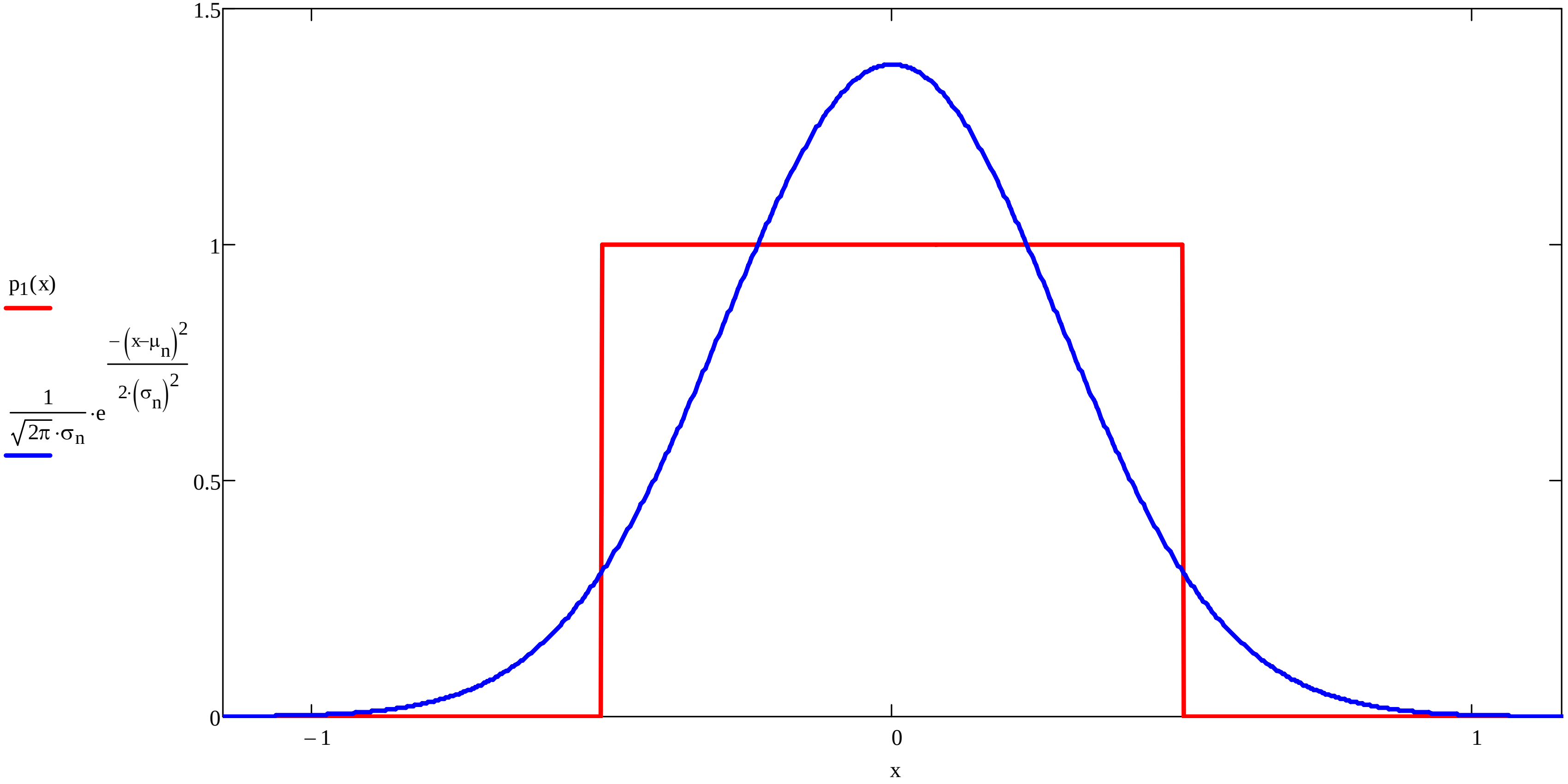
$$\rho = \lambda_A \text{(average service time)}$$

$$\rho = \text{average number in service facility} = \lambda_A / \mu$$

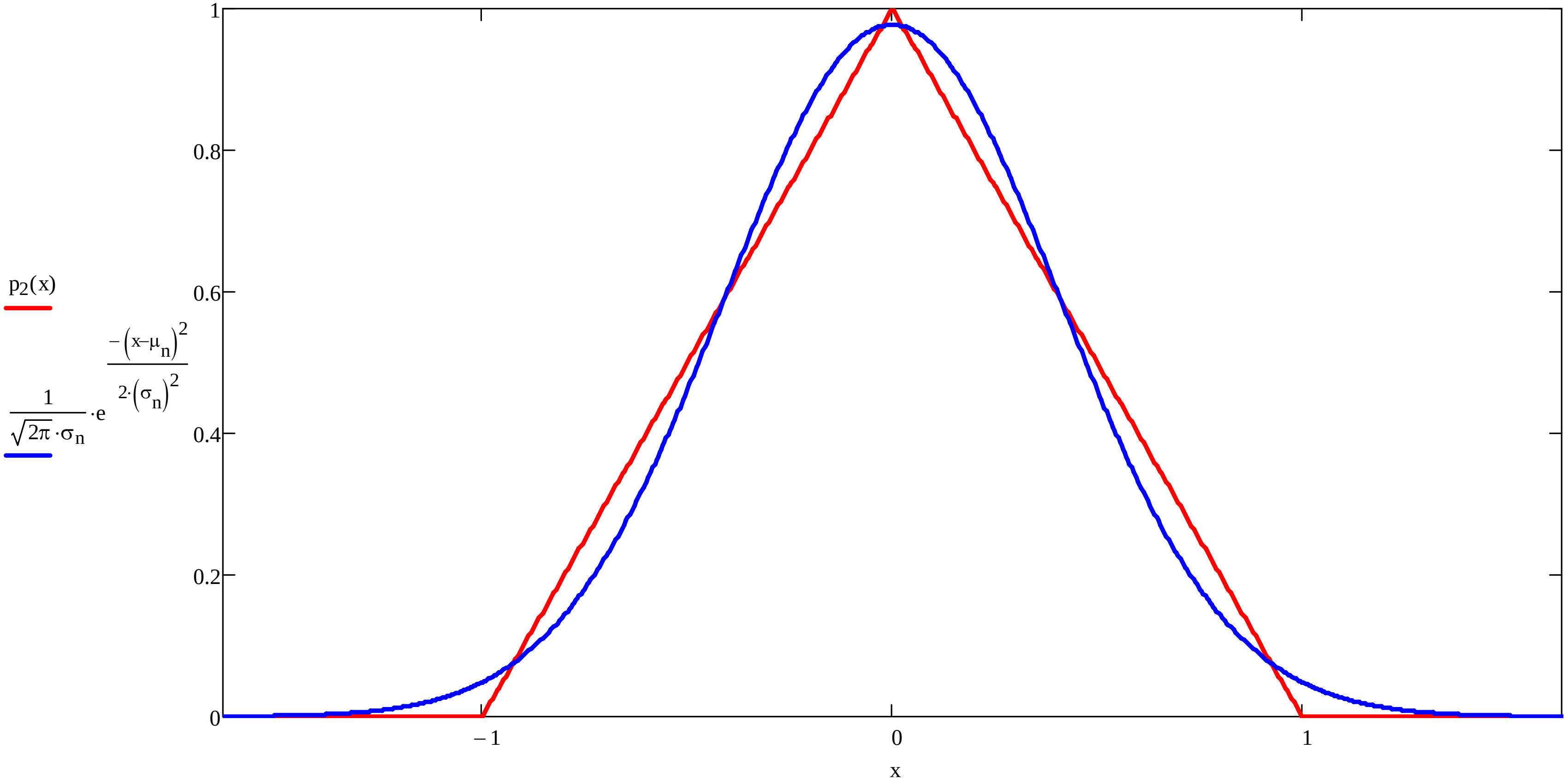
$$\lambda_A = \mu\rho = \mu(1 - e^{-\lambda/\mu})$$

$$W = \frac{L}{\lambda_A} = \frac{\lambda/\mu}{\mu(1 - e^{-\lambda/\mu})} = \frac{\lambda}{\mu^2(1 - e^{-\lambda/\mu})}$$

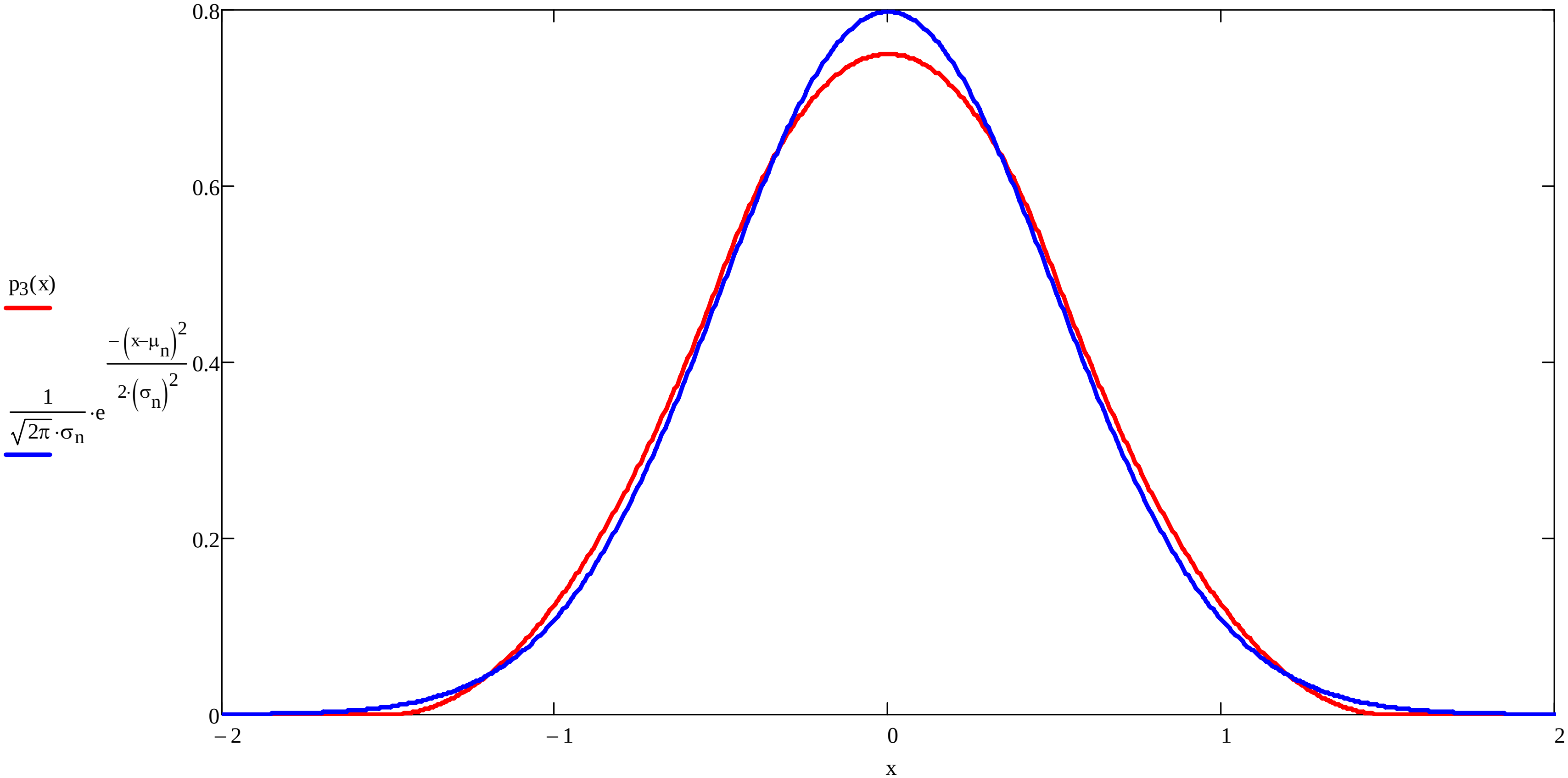
Central Limit Theorem Demo
Thanks to Prof. Dan Frey! :)



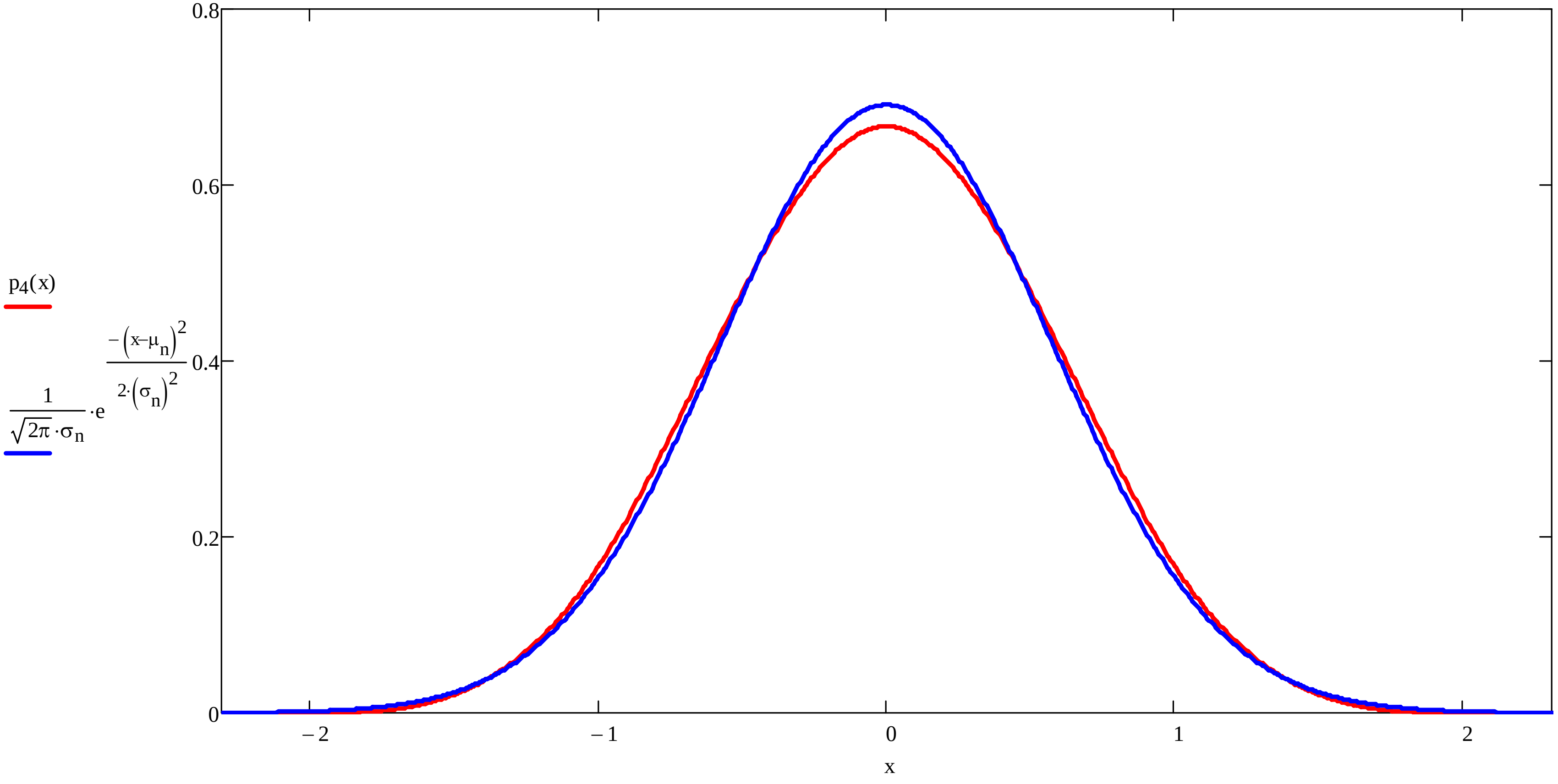
One Uniformly Distributed Random Variable



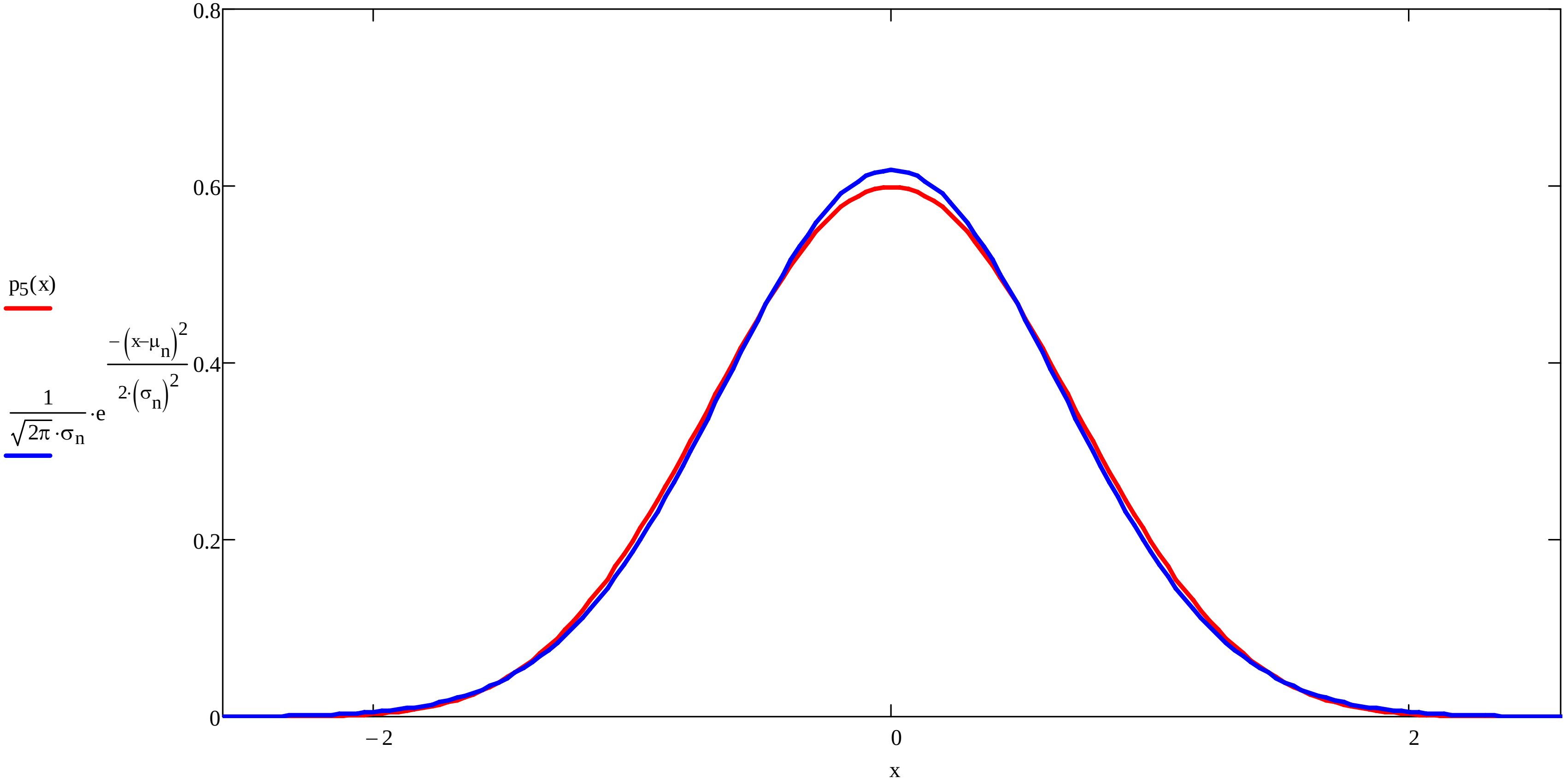
Sum of 2 iid Uniformly Distributed Random Variables



Sum of 3 iid Uniformly Distributed Random Variables



Sum of 4 iid Uniformly Distributed Random Variables



Sum of 5 iid Uniformly Distributed Random Variables

It’s Movie Time!

The Gaussian or Normal PDF

$$f_Y(y) = \frac{1}{\sigma_Y \sqrt{2\pi}} e^{-\{(y-E[y])^2 / (2\sigma_Y^2)\}} \qquad -\infty < y < \infty$$

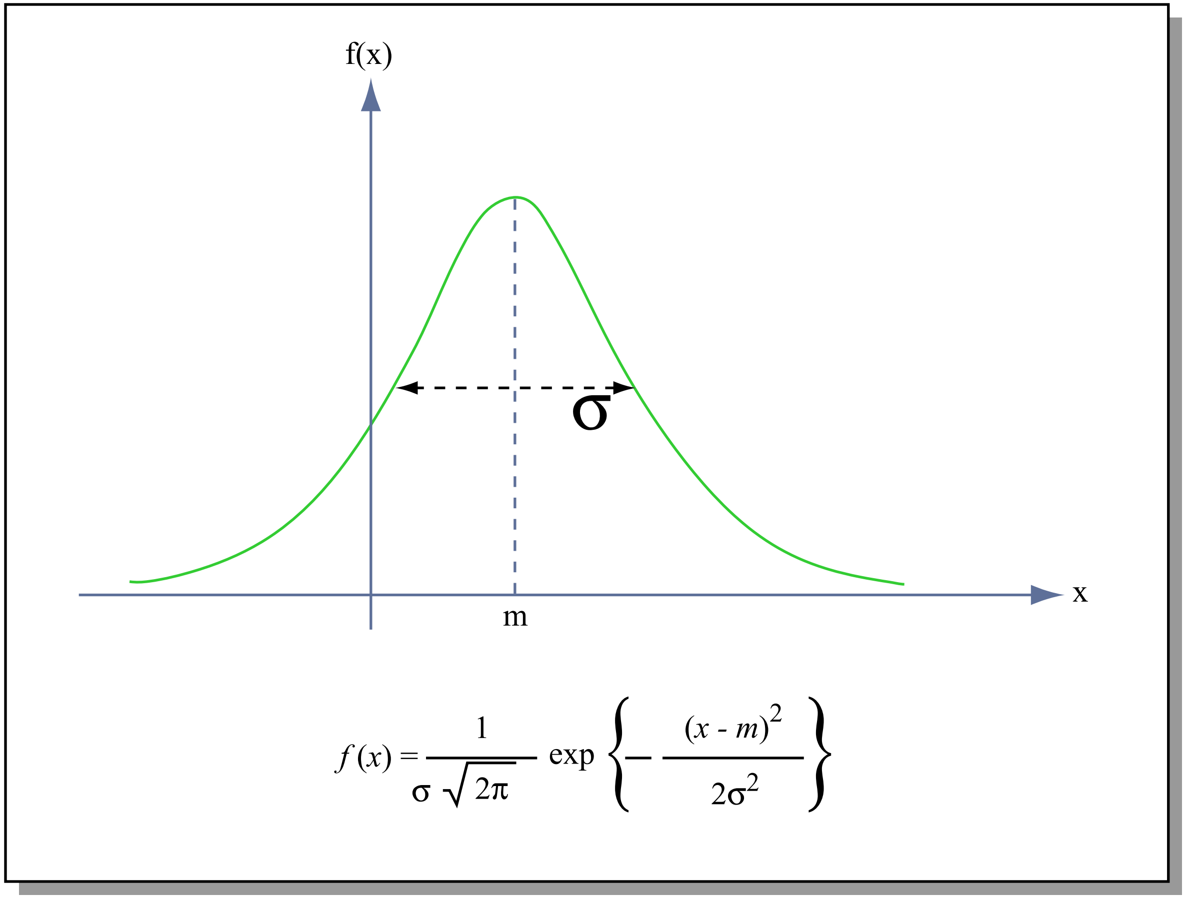


Figure by MIT OCW.

http://coding.yonsei.ac.kr/images/f10.gif

Central Limit Theorem

- Consider the sum S_n of n iid random variables X_i, where

$$E[X_i] = m_X < \infty$$

$$VAR[X_i] = \sigma_X^2 < \infty$$

$$S_n = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i$$

- Then, as n "gets large," S_n tends to a Gaussian or Normal distribution with mean equal to nm_X and variance equal to $n\sigma_X^2$.

$$S_n = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i$$

$$f_{S_n}(y) = \frac{1}{\sigma_X \sqrt{2\pi n}} e^{-\{(y - nm_X)^2 / (2n\sigma_X^2)\}} \qquad -\infty < y < \infty$$

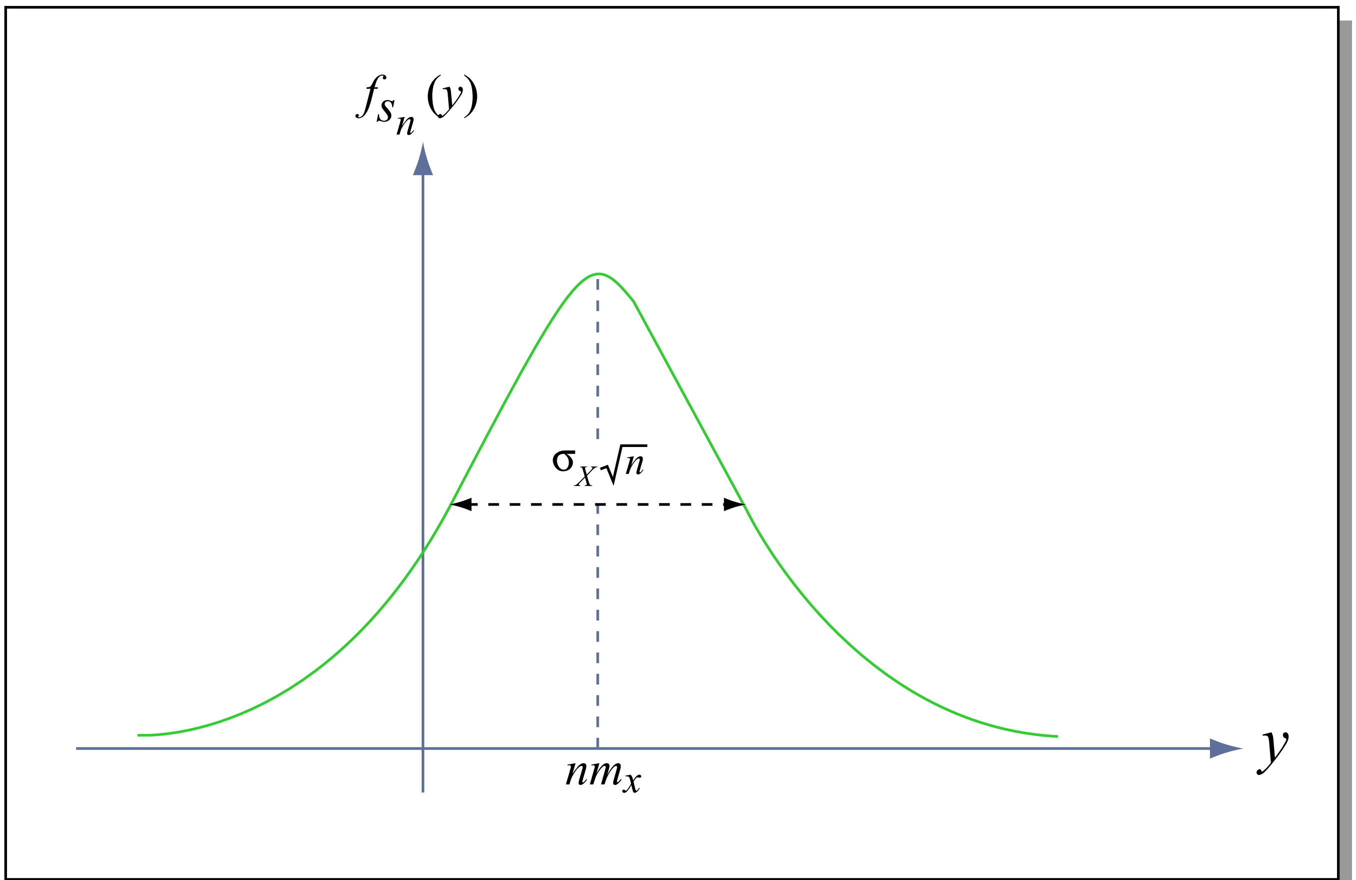


Figure by MIT OCW.

Normalizing Random Variables

Suppose we have a r.v. W having

$\text{Mean} = E[W] = a$ and

$\text{Variance} = E[(W-a)^2] = \sigma_W^2$

Define a new r.v.

$X \equiv W - a.$ Then

$$E[X] = E[W-a] = E[W] - a = a - a = 0$$

$$VAR[X] = VAR[W] = \sigma_W^2$$

Normalizing Random Variables

Suppose we have a r.v. W having

$\text{Mean} = E[W] = a$ and

$\text{Variance} = E[(W-a)^2] = \sigma_W^2$

Define a new r.v.

$X \equiv W - a$. Then

$E[X] = E[W-a] = E[W] - a = a - a = 0$

$VAR[X] = VAR[W] = \sigma_W^2$

Or suppose we define

$Y \equiv \gamma W$. Then

$E[Y] = \gamma E[W] = \gamma a$

$\sigma_Y^2 = E[(\gamma W - \gamma a)^2] = \gamma^2 E[(W-a)^2] = \gamma^2 \sigma_W^2$

Normalizing Random Variables

Suppose we have a r.v. W having

$$\text{Mean} = E[W] = a \quad \text{and}$$

$$\text{Variance} = E[(W - a)^2] = \sigma_W^2$$

Thus, if we define

$$Z \equiv (W - a)/\sigma_W, \text{ then}$$

$$E[Z] = 0$$

$$\sigma_Z^2 = 1$$

Z is called a normalized r.v.

Most table lookups of the Gaussian are via the CDF, with a normalized r.v.

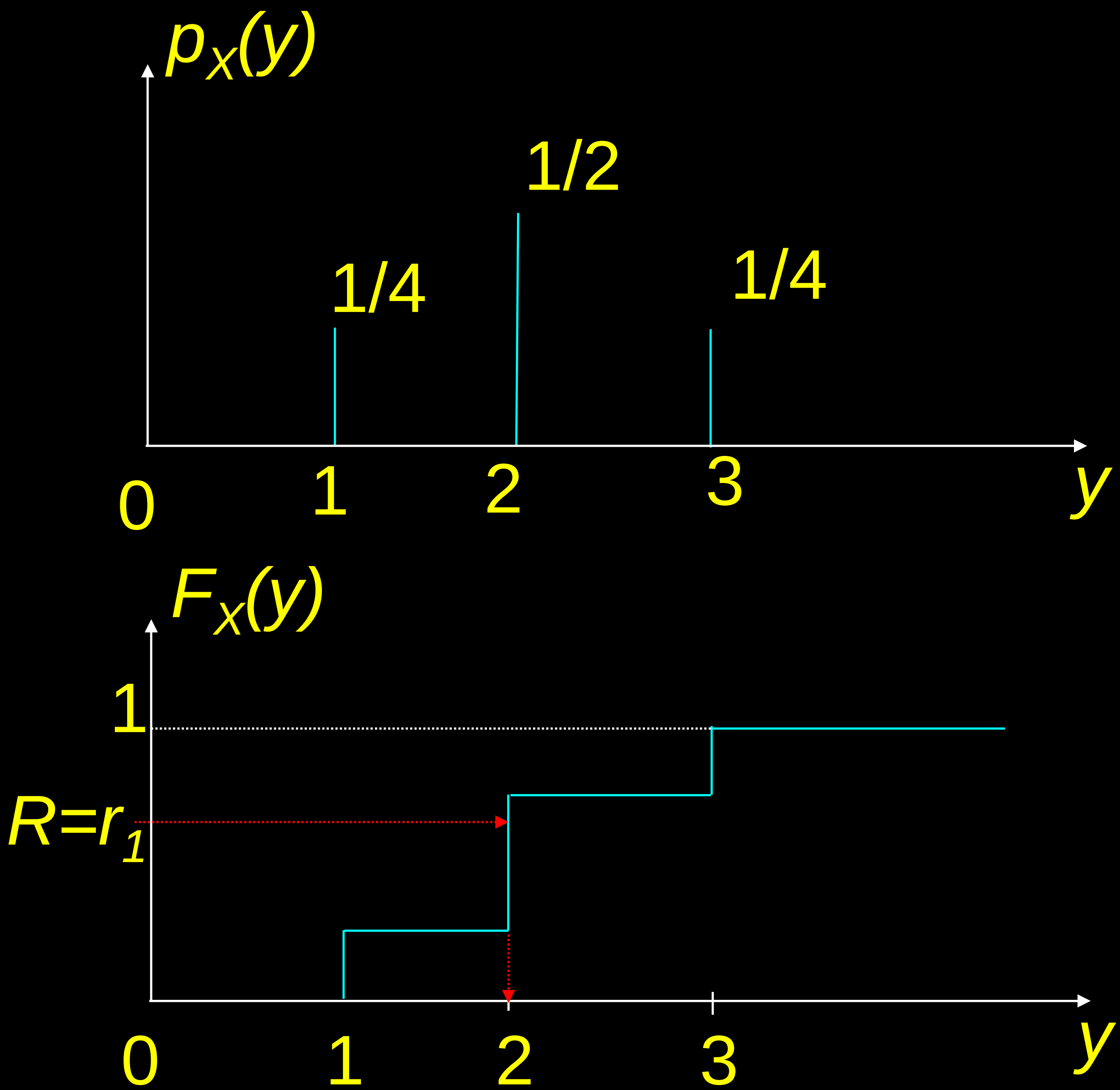
Obtaining Samples of the Gaussian R.V.

In Monte Carlo simulations, one often uses the Central Limit Theorem (CLT) to approximate the Gaussian.

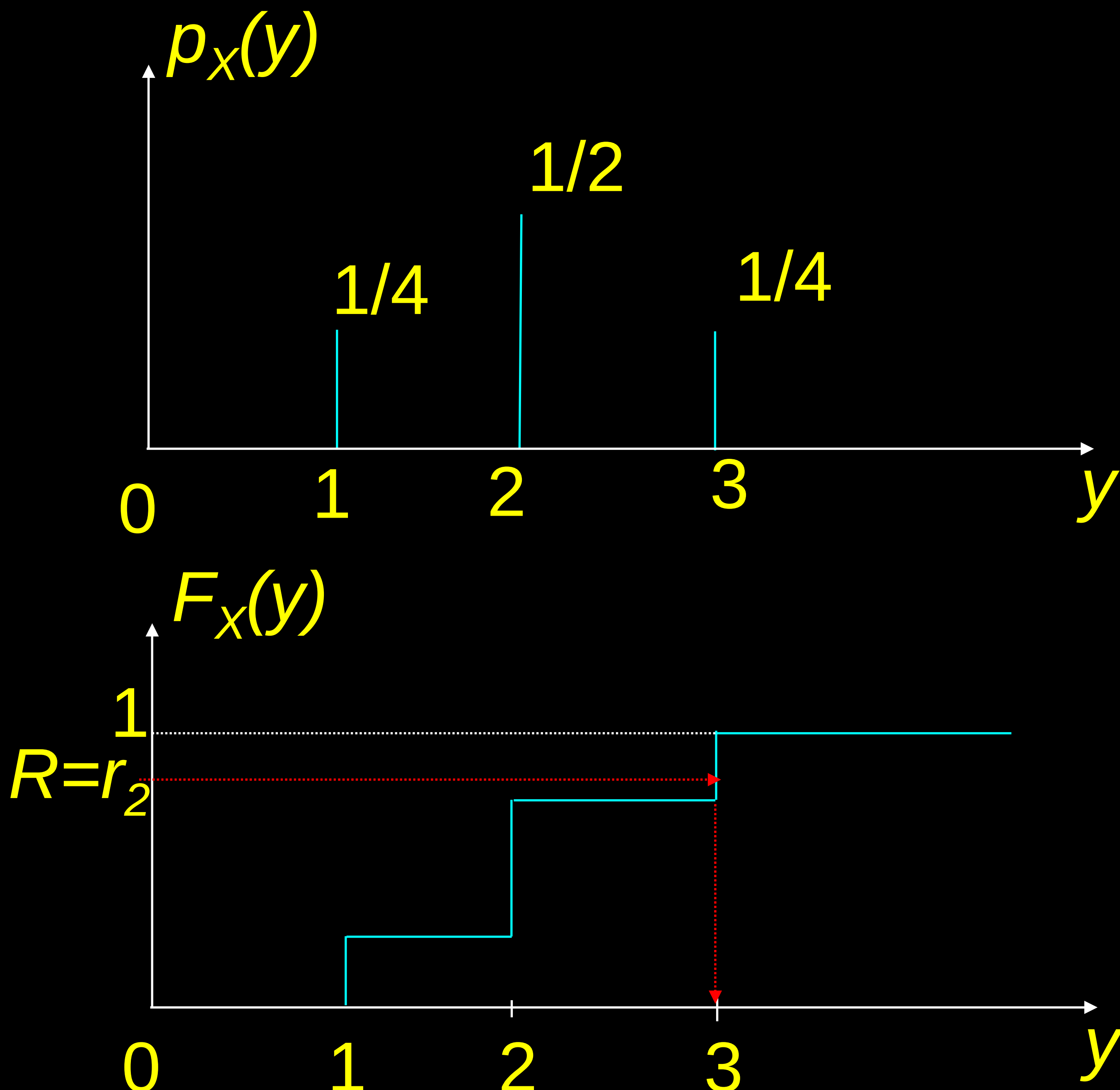
Example 1: Erlang Order *N* for large *N* should be approximately "Gaussian"

Example 2: Sum and normalize 12 uniforms over [0,1]. Good idea?

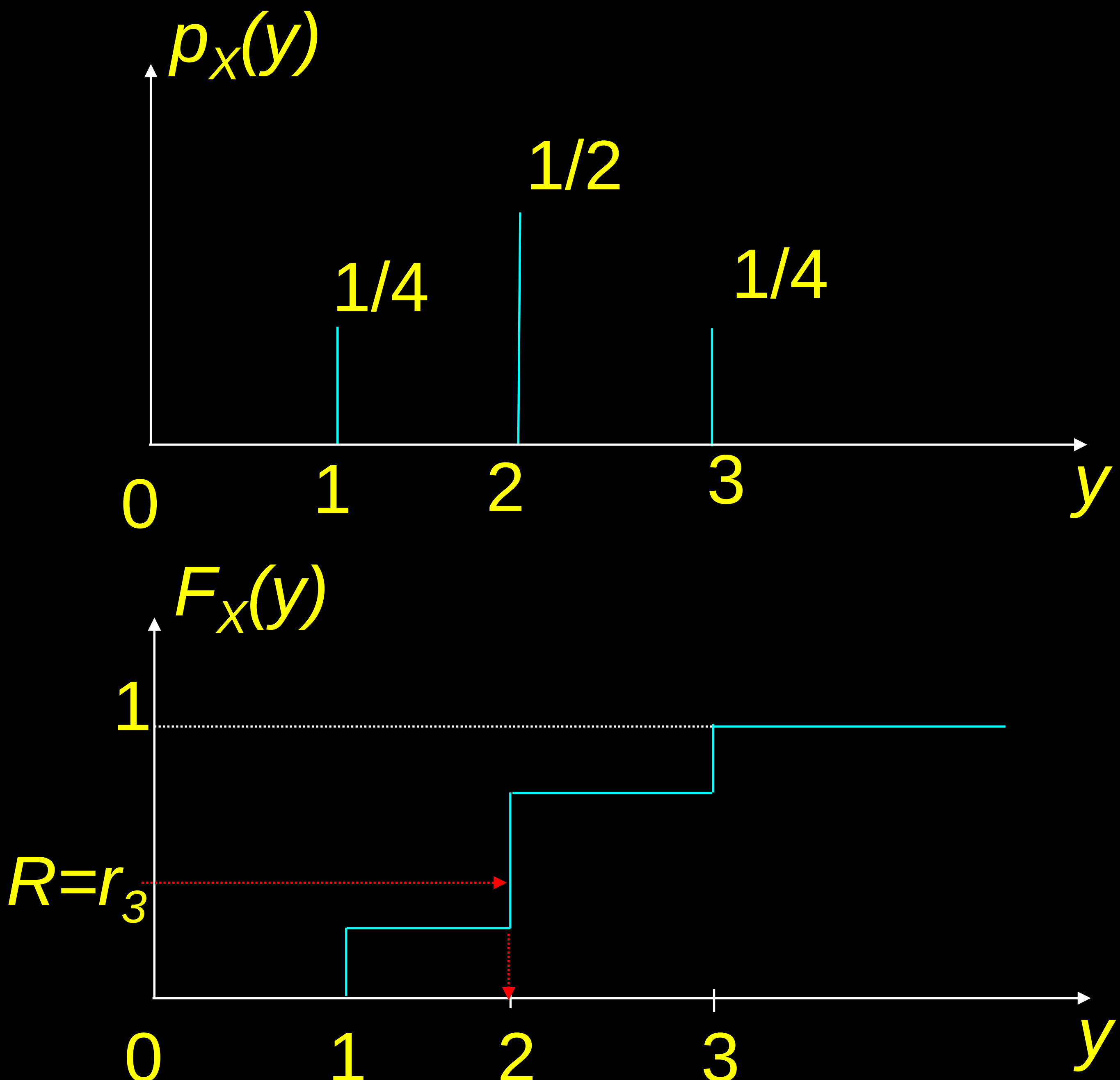
Let's talk about Monte Carlo sampling: **Inverse Method.** Uses CDF, and is Never Fail!



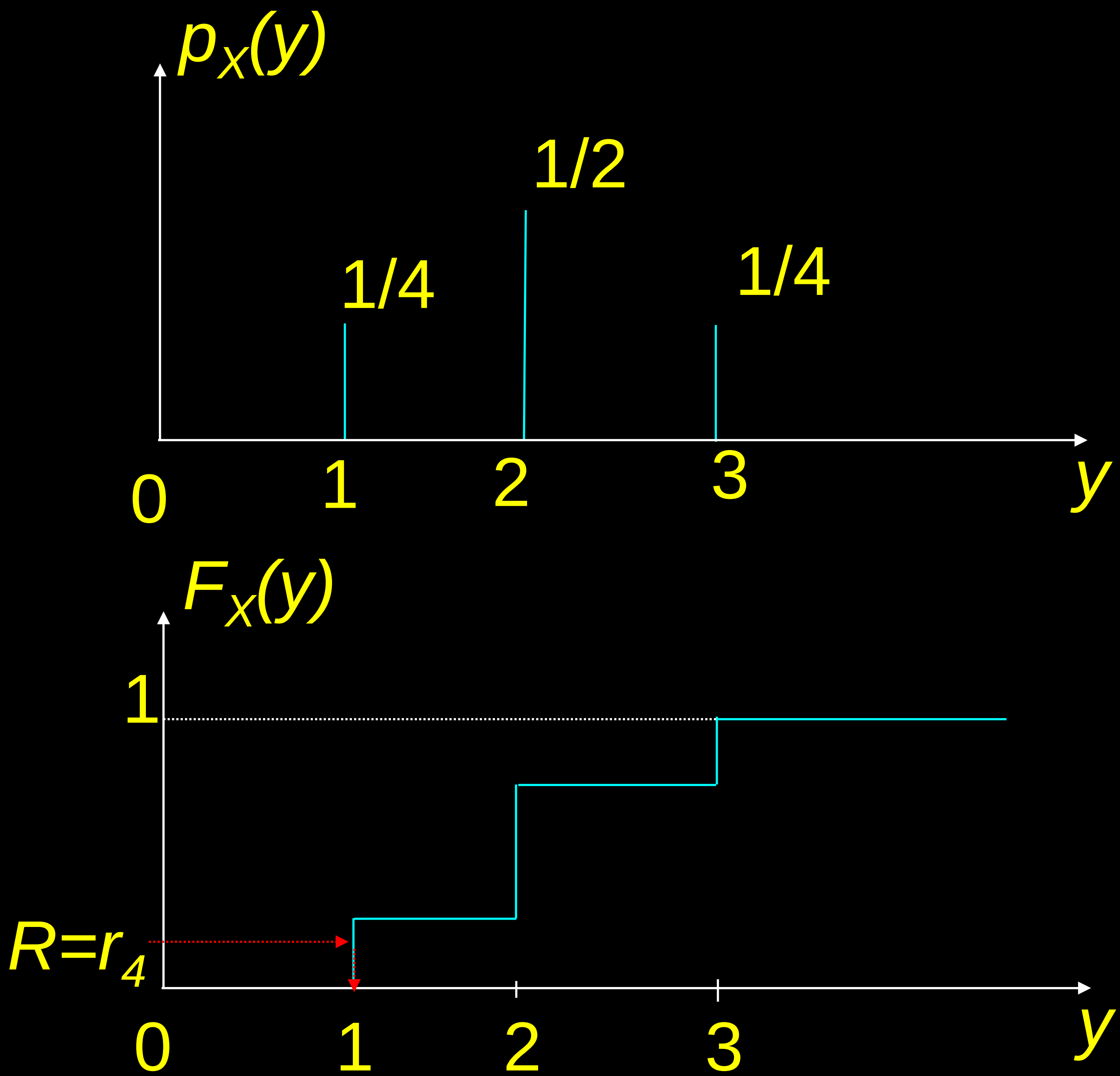
$p_X(y)$
1/4
1/2
1/4
0
1
2
3
y
$F_X(y)$
1
$R=r_1$
0
1
2
3
y



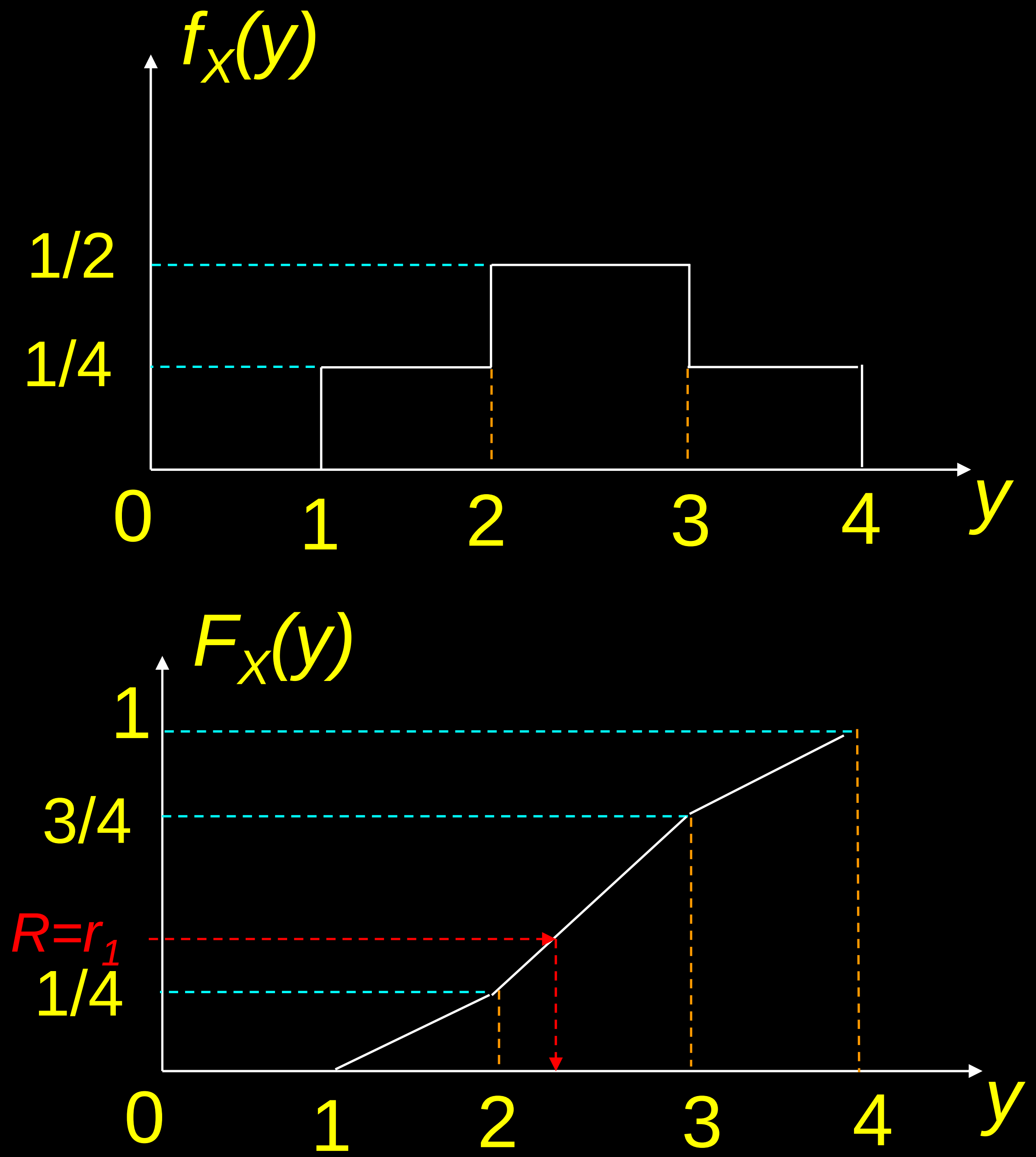
$p_X(y)$
1/4
1/2
1/4
0
1
2
3
y
$F_X(y)$
1
$R=r_2$
0
1
2
3
y



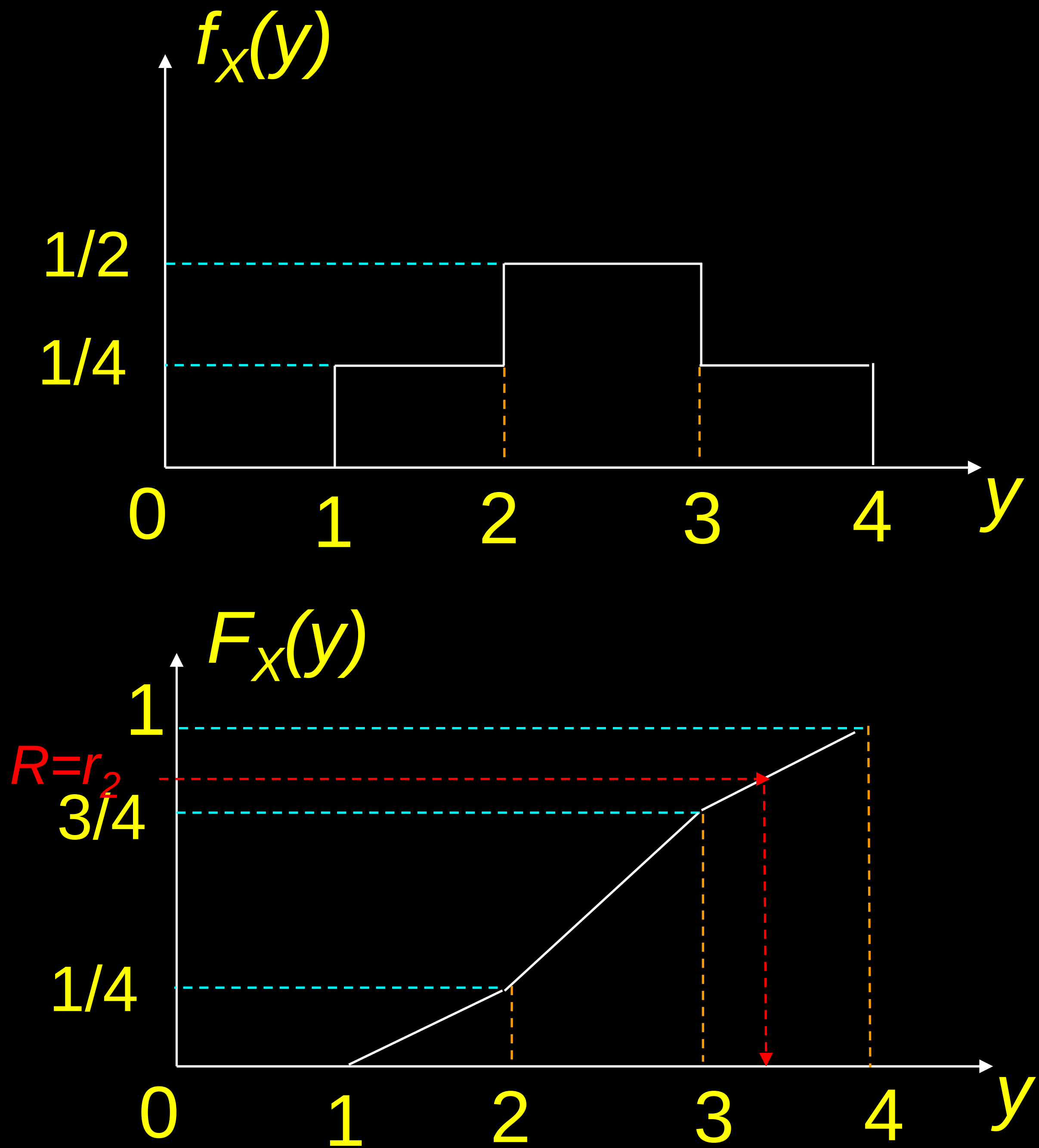
$p_X(y)$
1/4
1/2
1/4
0
1
2
3
y
$F_X(y)$
1
$R=r_3$
0
1
2
3
y

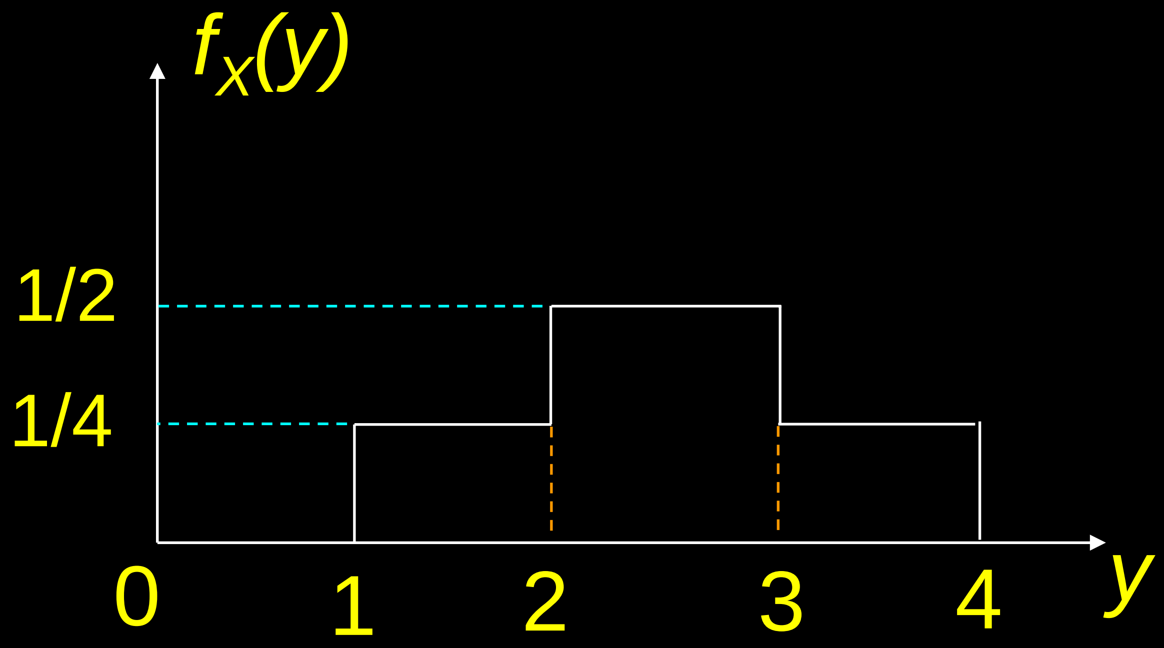

$p_X(y)$
1/2
1/4
1/4
0
1
2
3
y
$F_X(y)$
1
$R=r_4$
0
1
2
3
y

Inverse Method Also Works for Continuous Random Variables

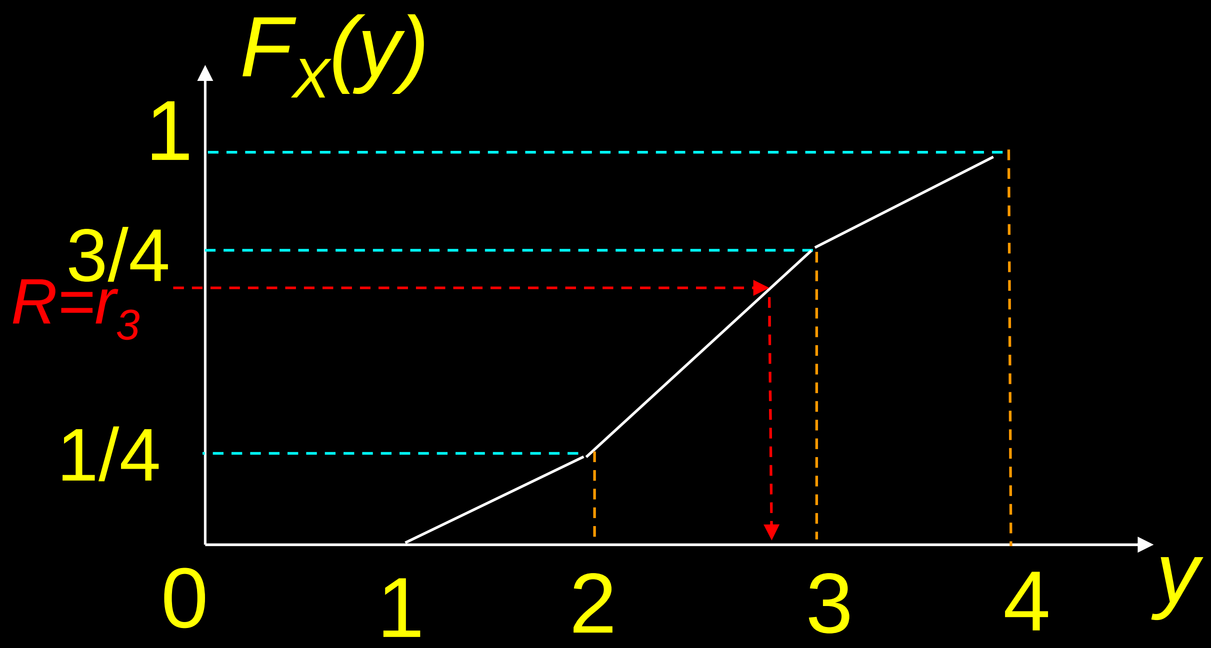


fX(y)
1/2
1/4
0
1
2
3
4
y
FX(y)
1
3/4
R=r1
1/4
0
1
2
3
4
y

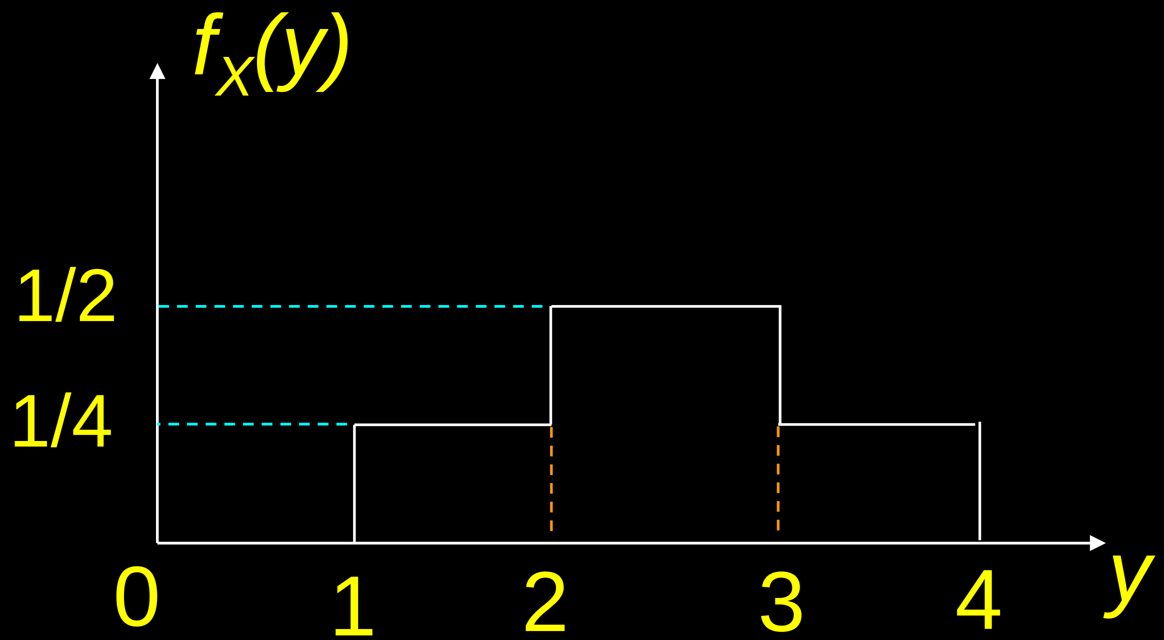

fX(y)
1/2
1/4
0
1
2
3
4
y
FX(y)
1
R=r2
3/4
1/4
0
1
2
3
4
y



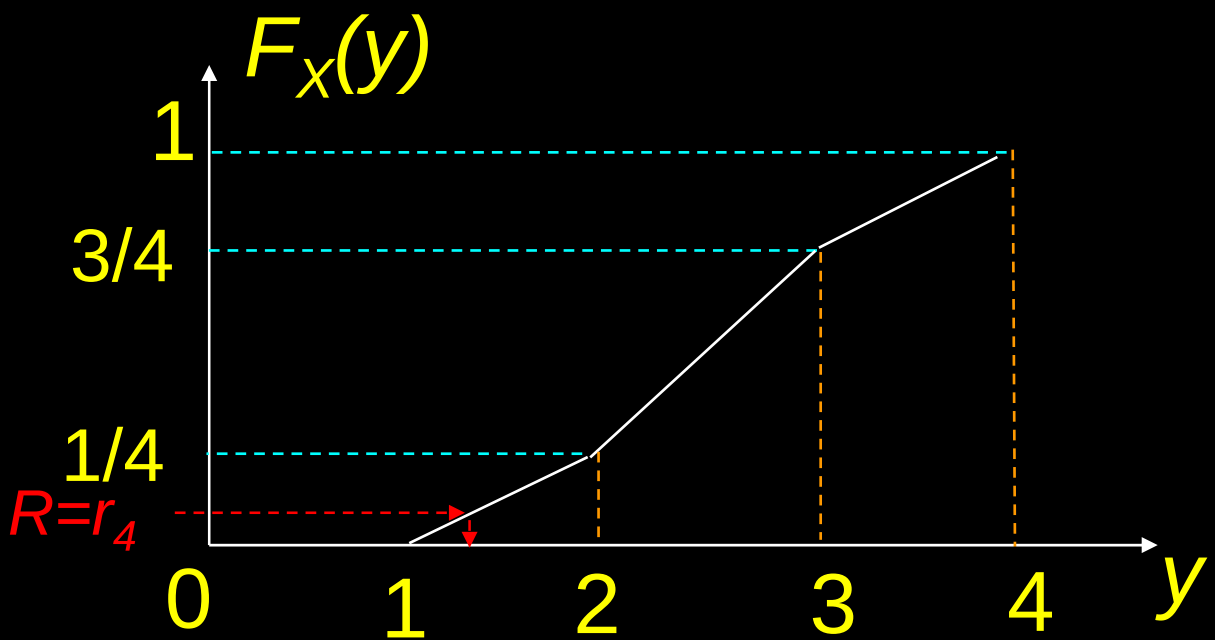
fX(y)
1/2
1/4
0
1
2
3
4
y



FX(y)
1
3/4
R=r3
1/4
0
1
2
3
4
y



fX(y)
1/2
1/4
0
1
2
3
4
y



FX(y)
1
3/4
1/4
R=r4
0
1
2
3
4
y

Time to Buckle your Seatbelts!

http://www.census.gov/pubinfo/www/multimedia/img/seatbelt-lo.jpg

Example 3: The "Relationships Method"

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \quad -\infty < x < \infty$$

X and Y are zero-mean independent Gaussian r.v.'s.

$$R \equiv \sqrt{X^2 + Y^2}$$

$$F_R(r) \equiv P\{R \le r\} = P\{\sqrt{X^2 + Y^2} \le r\}$$

$$F_R(r) = \int\int \frac{1}{2\pi\sigma^2} e^{-(x+y)^2/2\sigma^2} dxdy$$

circle of

radius r

$$F_R(r) = \int\int \frac{1}{2\pi\sigma^2} e^{-(x+y)^2/2\sigma^2} dxdy$$

circle of

radius r

$$f_R(\rho)d\rho = \int_{\theta=0}^{2\pi} d\theta \rho d\rho \frac{1}{2\pi\sigma^2} e^{-\rho^2/2\sigma^2} = \frac{\rho}{\sigma^2} e^{-\rho^2/2\sigma^2} d\rho, \quad \rho \geq 0$$

$$f_R(\rho) = \frac{\rho}{\sigma^2} e^{-\rho^2/2\sigma^2}, \quad \rho \geq 0$$

A Rayleigh pdf
With parameter $1/\sigma$

$$F_R(\rho) \equiv P\{R \le \rho\} = 1 - e^{-\rho^2/2\sigma^2}, \quad \rho \ge 0$$

$R_1 \equiv$ sample from a uniform pdf over [0,1]

$R_1 = 1 - e^{-\rho^2/2\sigma^2}$, which implies that

$$\rho = \sigma\sqrt{-2\ln(1-R_1)}$$

$$\theta = 2\pi R_2$$

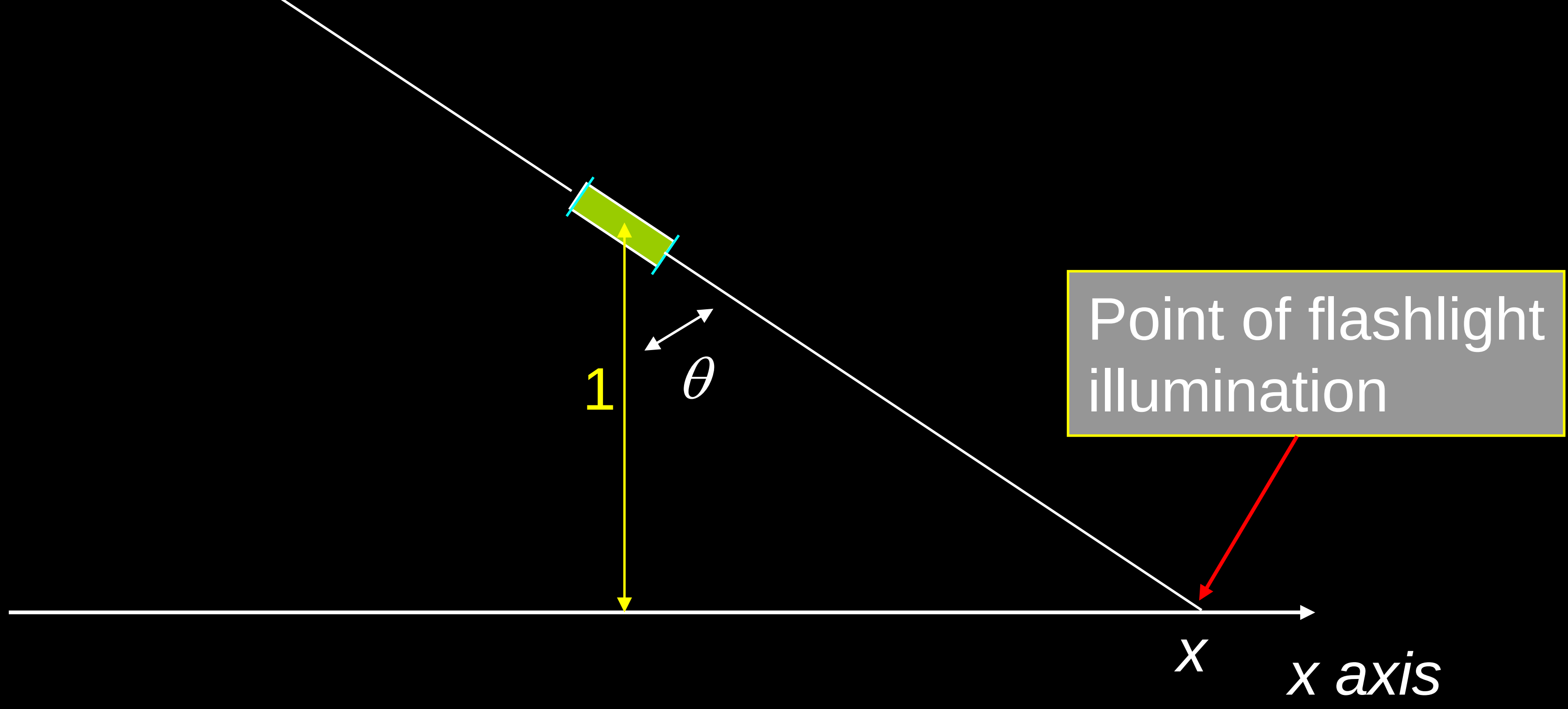
$$X = \rho\cos\theta = \sigma\sqrt{-2\ln(1-R_1)}\cos(2\pi R_2)$$

$$Y = \rho\sin\theta = \sigma\sqrt{-2\ln(1-R_1)}\sin(2\pi R_2)$$

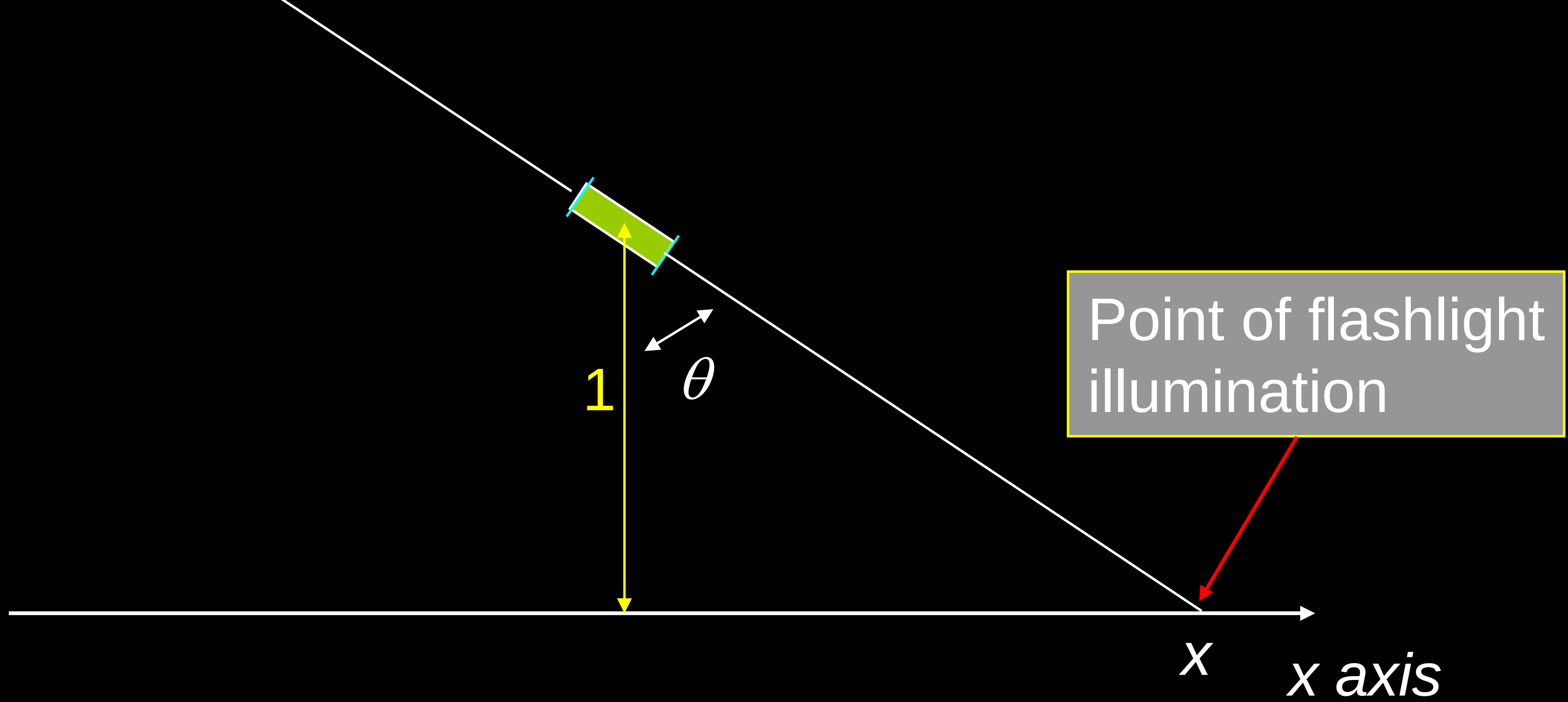
Here we have 2 exact samples from the Gaussian pdf, with no approximation from the CLT!

Spin the Flashlight

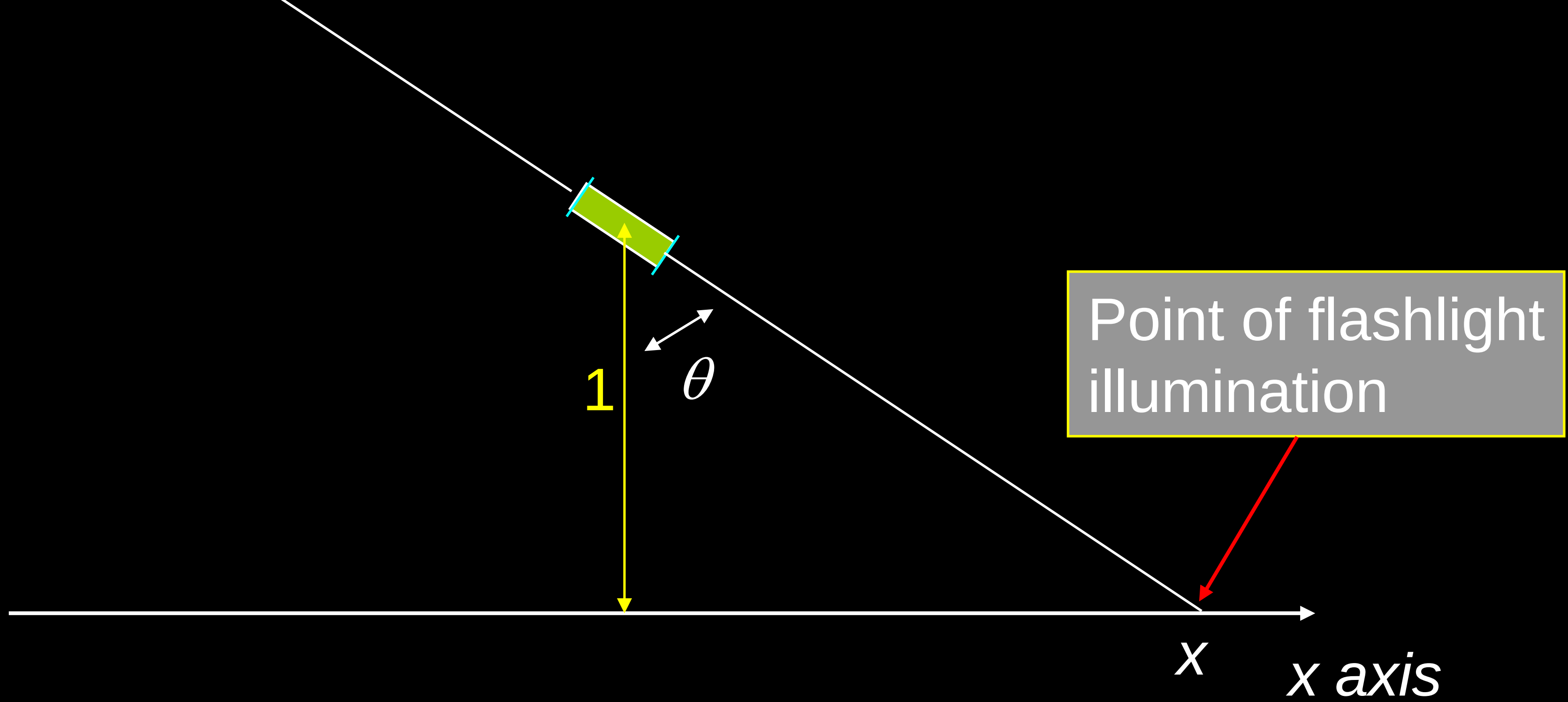
And, so, finite variance is just a professor's oral exam trick question? :)



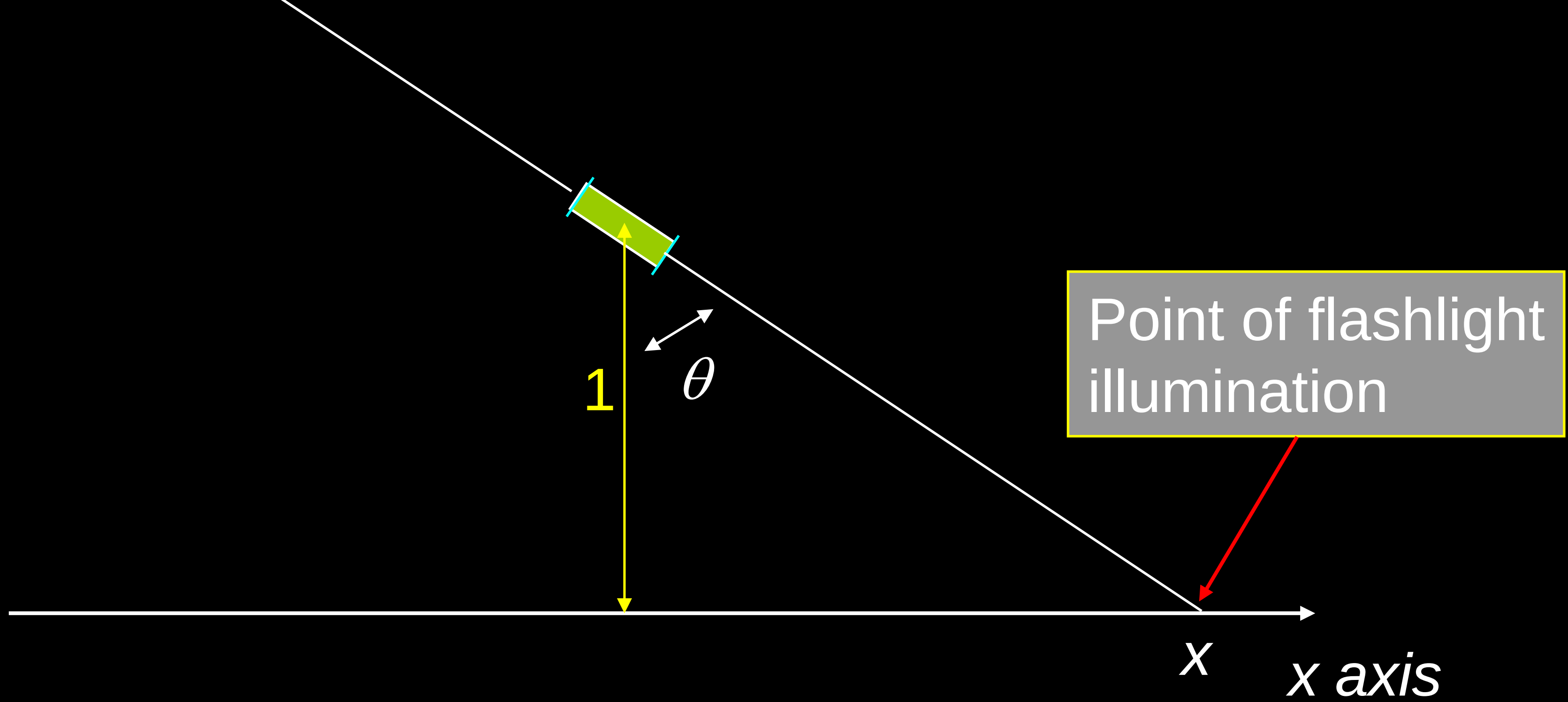
Point of flashlight illumination
1
θ
x
x axis



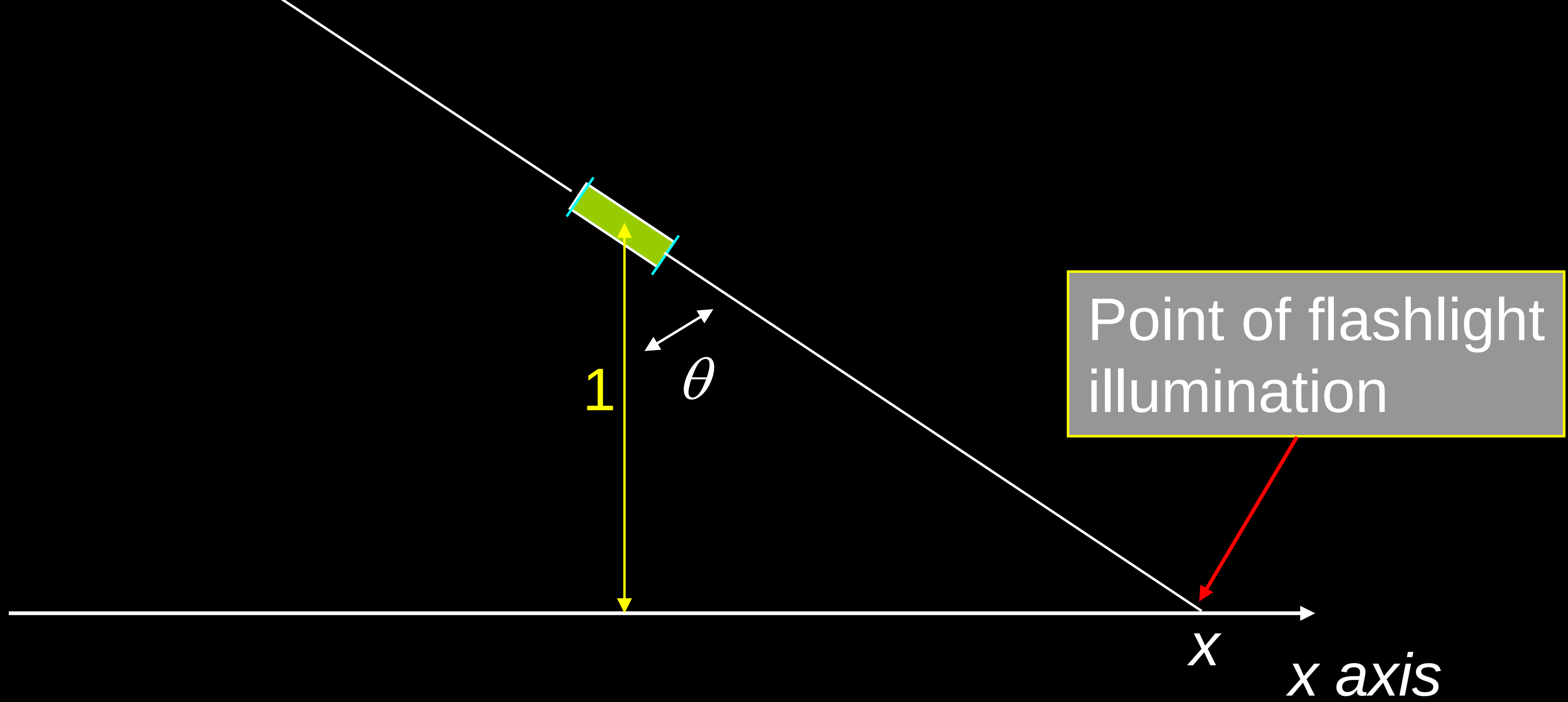
1. R.V.'s: X, Θ



1. R.V.': *X*, *Θ*
2. Sample space for *Θ*: [-π/2, π/2]



1. R.V.': X, Θ
2. Sample space for Θ: $[-\pi/2, \pi/2]$
3. Θ uniform over $[-\pi/2, \pi/2]$



1. R.V.': X, Θ
2. Sample space for Θ: $[-\pi/2, \pi/2]$
3. Θ uniform over $[-\pi/2, \pi/2]$
4. (a) $F_X(x) = P\{X<x\} = P\{\tan\Theta<x\} = P\{\Theta < \tan^{-1}(x)\} = 1/2 + (1/\pi)\tan^{-1}(x)$

   (b) $f_X(x) = (d/dx)\, F_X(x) = 1/(\pi)(1 + x^2)$ all x

Cauchy pdf

Mean =?, Variance = ????