

## Problem Set #2

### 1. Network robustness – 4 pts

#### a) 1 pt

Since each link can be functioning or broken, we have a total of  $2^5 = 32$  possibilities that will define the sample space.

If we define 0: broken state; 1: functioning state and if we use the following order to characterize the state of each link of the network: ABCDE, we will have the following sample space:

00000; 00001; 00010; 00011; 00100; 00101; 00110; 00111; 01000; 01001; 01010; 01011; 01100; 01101; 01110; 01111; 10000; 10001; 10010; 10011; 10100; 10101; 10110; 10111; 11000; 11001; 11010; 11011; 11100; 11101; 11110; 11111.

#### b) 1 pt

We can generalize: the size of the sample space of a similar network with n links will be  $2^n$ . The size of the faulty sub-space will depend upon the topology of the network.

#### c) 1 pt

There is two possible understandings of this question. Since the question was ambiguous, I've counted both of them as correct.

If we consider that each line fails independently from the corresponding ones in the other sub-networks, the answer is obvious:

$$P[T_{Sni}] = P_{E_{Sni}} + (1 - P_{E_{Sni}}) \left( \left[ P_{A_{Sni}} + (1 - P_{A_{Sni}}) \times P_{D_{Sni}} \right] \times \left[ P_{B_{Sni}} + (1 - P_{B_{Sni}}) \times P_{C_{Sni}} \right] \right)$$

$$P[T_{1-2}] = \prod_{i=1}^3 P[T_{Sni}]$$

If all corresponding lines in the sub-networks (e.g. all lines  $P_{B, Sni}$ ) fail at the same time, then the reliability of the network will be equal to the reliability of the sub-network made with the less reliable component lines of each type:

$$P[T_{1-2}] = \text{Min}_{Sni}[P_{E_{Sni}}] + (1 - \text{Min}_{Sni}[P_{E_{Sni}}]) \left( \left[ \text{Min}_{Sni}[P_{A_{Sni}}] + (1 - \text{Min}_{Sni}[P_{A_{Sni}}]) \times \text{Min}_{Sni}[P_{D_{Sni}}] \right] \times \left[ \text{Min}_{Sni}[P_{B_{Sni}}] + (1 - \text{Min}_{Sni}[P_{B_{Sni}}]) \times \text{Min}_{Sni}[P_{C_{Sni}}] \right] \right)$$

#### d) 1 pt

The probability of successful transmission  $P_{2i}$  of a link constituted by 2 parallel components having an individual probability of successful transmission  $P_i$  is:

$$P_{2i} = 2 P_i - P_i^2$$

Then, we have to successively replace the probabilities of successful transmission of all links of the network composed by the three subnets by the probability of successful transmission of a double link.

The optimal solution is the one that maximizes the global probability of successful transmission. When the network becomes too complicated to test all possibilities, optimization algorithms can be developed to find the best solution.

**2. Binomial distribution and baseball – 3 pts**

a) 1 pt

$$P[X]: P[X] = \begin{cases} 0.4 & \text{iff } x = 0 \\ 0.6 & \text{iff } x = 1 \end{cases}$$

$$\text{Then, } P_X^T[z] = \sum_{i=0}^{\infty} z^x \cdot p_X(x) = z^0 \times 0.4 + z^1 \times 0.6$$

QED

b) 1 pt

$$N_w = \sum_{i=1}^{162} X_i$$

This is a series of independent Bernoulli trials with an identical success probability.

$$P_{N_w}^T[z] = \prod_{k=1}^{162} P_X^T[z] = \prod_{k=1}^{162} (0.4 + 0.6 \times z) = (0.4 + 0.6 \times z)^{162}$$

c) 1 pt

$$P_k^T = (0.4 + 0.6 \times z)^{162} = \sum_{k=0}^{162} \binom{162}{k} \times 0.4^{162-k} \times (0.6z)^k$$

We have:

$$P_k^T = \sum_{k=0}^{\infty} z^k P_k(k) = \sum_{k=0}^{162} z^k P_k(k)$$

$$\text{Therefore, } P_k = \binom{162}{k} \times 0.4^{162-k} \times 0.6^k$$

**3. Interviewing movie goers – 3 pts**

Psychologically speaking, people tend to go and see movie that others like to see. Therefore, successful movies will attract large audience, whereas unsuccessful ones will be watched by very few people.

This creates a selection bias. A simple example will illustrate it. Let's suppose that all movie theaters rooms have 100 seats. Let's assume rooms projecting a successful movie will be filled at 67% and that rooms projecting an unsuccessful one will be filled at 1%. Finally, we'll make the assumption that we have on average 1 successful movie for 15 unsuccessful ones.

Then, the apparent fill rate is:

$$F_{App} = \frac{67}{100} \times \frac{67}{82} + \frac{1}{100} \times \frac{15}{82} = 0.549$$

But the real fill rate is in fact:

$$F_{Real} = \frac{67 + 15}{100 + 15 \times 100} = 0.051$$

If we make a two decimals approximation, we obtain respectively 55% and 5%.

#### 4. Jogging – 4 pts

##### a) 2 pts

Four equally likely possibilities when you pass another jogger:

- You are both on your way back
- You are one your way back, but he hasn't yet turned around
- You haven't yet turned around, but he is on his way back
- You both haven't yet turned around

Only in the last case have you a chance to meet him again. The probability of meeting him again is thus inferior to 25%.

##### b) 2 pts

4 possible cases (each with probability 25%):

1) Jogger enters the path in front of you and go in the same direction as you): Since everybody moves at the same speed, you will pass him if you jog at least 4 units of distance before turning around. You have decided to jog  $X$  units of distance before turning around, where  $P(X \leq x) = 1 - e^{-x}$ .

Therefore, the probability to pass him is  $P_{\text{Pass}} = e^{-4} \approx 0.0183$ .

2) Jogger enters the path in front of you and go in the opposite direction as you: You will pass him (once or more) if you jog at least 1 unit of distance before turning around.

Similarly, we find  $P_{\text{Pass}} = e^{-1} \approx 0.3679$ .

3) Jogger enters the path on your back and go in the same direction as you: You will pass him if you jog between 1 and 2 units of distance.

Thus, we have  $P_{\text{Pass}} = e^{-1} - e^{-2} \approx 0.2325$

4) Jogger enters the path on your back and go in the direction opposite to you: you will never pass him.

The probability of passing the other jogger at least once is:

$$P_{\text{Pass}} = 0.25 \times (e^{-4} + e^{-1} + e^{-1} - e^{-2}) \approx 0.1547$$

#### 5. Returning to the broken stick experiment – 6 pts

##### a) 1pt

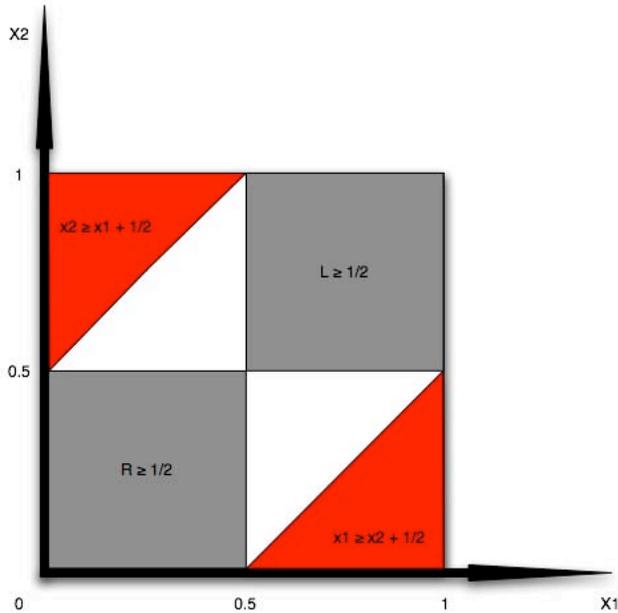
Let's consider the first case:  $P[L \geq 1/2]$ :  $P\{\text{one point is not in the left part of the yardstick}\} = 1/2$

Thus,  $P[L \geq 1/2] = (1/2)^2 = 1/4$

Similarly, we find  $P[R \geq 1/2] = (1/2)^2 = 1/4$

What about  $P[M \geq 1/2]$ ? Let  $X_1$  and  $X_2$  be the two points where the stick is cut.

Without loss of generality, let  $x_1$  be the abscissa of the leftmost point and  $x_2$  the abscissa of the rightmost one. Let's consider the sample space:



The probability distributions of the abscissas of  $X_1$  and  $X_2$  are uniformly distributed over the sample space. By observation, we can conclude that  $P[M \geq 1/2] = 1/4$ .

**b) 1 pt**

It suffices that one piece of the yardstick be longer than  $1/2$  yard to prevent a triangle to be formed. Besides, the probabilities that one given piece be longer than  $1/2$  yard are disjointed.

$$P\{\Delta\} \equiv P\{\text{a triangle can be formed from the 3 pieces}\}$$

Then,  $P\{\Delta\} = 1 - P\{[L \geq 1/2] \cup [M \geq 1/2] \cup [R \geq 1/2]\}$

**c) 1 pt**

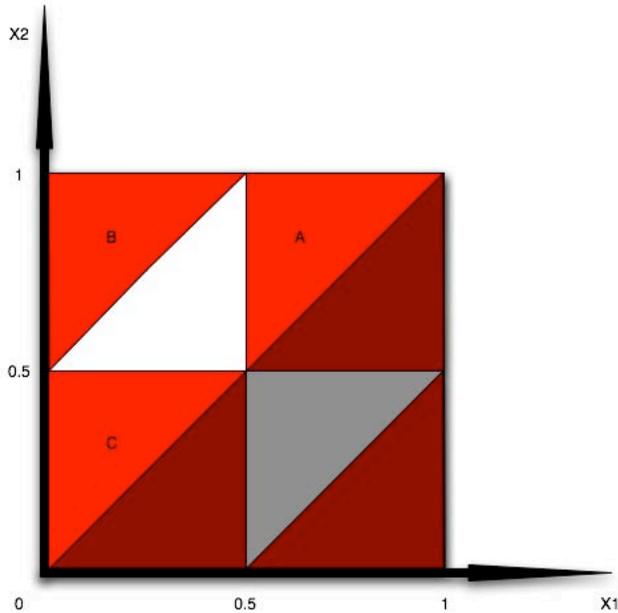
Let  $A = \{L \geq 1/2\}$ ;  $B = \{M \geq 1/2\}$ ;  $C = \{R \geq 1/2\}$ :

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \text{ and } P[A \cup B \cup C] = P[(A \cup B) \cup C]$$

$$P[(A \cup B) \cup C] = P[A \cup B] + P[C] - P[(A \cup B) \cap C]$$

$$\text{So, } P[A \cup B \cup C] = P[A] + P[B] - P[A \cap B] + P[C] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$$

**d) 1 pt**



Obviously, applying the results of question b) gives the correct answer.

e) **1 pt**

The probability that we cannot make a quadrilateral is equal to the probability that one of the four pieces is longer than the sum of all the others, i.e. that it is longer than half the initial yardstick.

By the same logic, the probability that one given piece is longer than  $\frac{1}{2}$  yard is equal to  $\frac{1}{8}$ .

Therefore,  $P[Q] = P\{a\_quadrilateral\_can\_be\_formed\} = 1 - 4 \times \frac{1}{8} = 0.5$

f) **1 pt**

For a  $n$ -gon, the yardstick is cut into  $n$  pieces. The probability that a  $n$ -gon can be formed is equal to the probability that one of the pieces is longer than  $\frac{1}{2}$  yard. The probability that one given piece be longer than  $\frac{1}{2}$  yard is  $\frac{1}{2^{n-1}}$ .

Thus,  $P[N] = P\{a\_n\_gon\_can\_be\_formed\} = 1 - n \times \frac{1}{2^{n-1}} = 1 - \frac{n}{2^{n-1}}$