16.888/ESD 77 Multidisciplinary System Design Optimization:
Assignment 5 Part a) Solution

Part A1

Recap from A4: This problem relates to A4 and the optimal solutions obtained by using SQP exhaustively for all possible discrete variable settings is shown in table below:

Problem formulation with SQP	b (m)	t (m)	h (m)	w (m)	d (m)	Beam material	Support Material	# beams	Optimal Cost
w/o shear stress constraint	0.3471	0.01	2.0	0.374	0.374	A514	Concrete	1	$5771.3
With constraint linking beam width (b), number of beams (N) and support depth (d)	0.3471	0.01	2.0	0.3664	0.3971	A514	Concrete	1	$5774.1

We will use the optimal vector thus obtained to answer all subsequent questions, wherever applicable.

(a) Here the applied load is given as: F = q*L and q (load per unit length) becomes an additional continuous design variable. For calculation of the Jacobian, we have to leave out the discrete variables like material id's and number of beams.
The Jacobian with respect to continuous design variables can be written as:

$$Jacobian = \begin{bmatrix} \frac{\partial C}{\partial X} & \frac{\partial F}{\partial X} \end{bmatrix}^T = \begin{bmatrix} 2At & 0 \\ A(2b+h-4t) & 0 \\ At & 0 \\ Bd & 0 \\ Bw & 0 \\ 0 & L \end{bmatrix}^T \quad \text{where } X = [b \;\; t \;\; h \;\; w \;\; d \;\; q]';\; A = c_b L \rho_b n_b;\; B = c_s \rho_s H$$

Using the given data, we have, A=213.3e+3 (\$/m^2) for beam and B=0.48e+03 (\$/m^2) for concrete column.

Note this analysis will also depend on your problem formulation in A4 and would be carried over while tackling this question.

Considering the formulation laid out in A4, the Jacobian for the case where no shear stress constraint and no constraint on internal dependency of beam flange width with support column depth (d) is given by:

$$Jacobian = \begin{bmatrix} 4266 & 0 \\ 566141 & 0 \\ 2133 & 0 \\ 163.2 & 0 \\ 180 & 0 \\ 0 & 30 \end{bmatrix}^T$$

For the other case where we consider shear stress constraint and also a constraint on internal dependency of beam flange width (b) and column depth(d), the Jacobian is:

$$Jacobian = \begin{bmatrix} 4266 & 0 \\ 566141 & 0 \\ 2133 & 0 \\ 191 & 0 \\ 176.4 & 0 \\ 0 & 30 \end{bmatrix}^T$$

Notice the sensitivity of cost to the beam thickness (t). Also notice the changes in sensitivity ordering between column width (w) and column depth (d) in these two cases.

Note: You can formulate/interpret this problem in various ways. The solution outlined considers two functions around the optimal cost design and does not look at the overall constrained problem. One can also think in terms of 5 design variables [b t h w d] and compute the sensitivity of F = qL with all constraints being active at the optimal point. This can be represented as, $q = \min\{q_1, q_2, q_3, q_4\}$ where q_i refers to i^{th} stress constraint and F = qL.

At the minimal cost design, the beam bending stress limit yields the limiting allowable load on the bridge. Therefore we can write q as:

$$q = \frac{1}{225}\left[\frac{112\times10^8 I_{beam}}{h} - 17.4373\times10^6\{2bt + (h-2t)t\}\right]$$
$$\text{where } I_{beam} = \frac{t}{12}(h-2t)^3 + 2\left[\frac{bt^3}{12} + bt\left(\frac{h}{2}-\frac{t}{2}\right)^2\right] \qquad (1)$$

Using (1) in F=qL and apply partial differentiation at the optimal point, we can write the Jacobian with 5 design variables {b t h w d} as (the row corresponding to cost does not change):

$$Jacobian = \begin{bmatrix} 4266 & 14.738\times10^{6} \\ 566141 & 970.2\times10^{6} \\ 2133 & 7.47\times10^{6} \\ 191 & 0 \\ 176.4 & 0 \end{bmatrix}^{T}$$

The constraining load intensity does not involve {w d} and therefore associated sensitivities are zero. Note that the Jacobian would change its structure if limiting load intensity was due to a different stress constraint.

(b) In this case of finding target isocost designs, formulation with constraints on shear stress and internal dependency between 'b' and 'd' was used and the discrete variables were help constant at theirs values at optimal design. The basic goal programming formulation can be written as:

$$\min_{X}\ \text{goal} = \|\text{Cost-1.1C}^{*}\|\ \text{ where } C^{*} = \text{the optimal cost}$$

$$\begin{aligned}
s.t\ & \frac{2t}{h} \le 1 \\
& \frac{t}{b} \le 1 \\
& \sigma_{bending} \le \sigma_{failure,bending} \\
& \sigma_{shear} \le \sigma_{failure,shear} \\
& \sigma_{\sup port} \le \sigma_{failure,\sup port} \\
& P_{applied} \le P_{critical,\sup port} \\
& d \ge b + 0.05 \\
& 0.1\text{ m} \le b \le 1\text{ m} \\
& 0.01\text{ m} \le t \le 0.5\text{ m} \\
& 0.1\text{ m} \le h \le 2\text{ m} \\
& 0.2\text{ m} \le w \le 2\text{ m} \\
& 0.3\text{ m} \le d \le 3\text{ m}
\end{aligned}$$

The table below shows two such isocost designs:

b (m)	**t (m)**	**h (m)**	**w (m)**	**d (m)**	**Beam material**	**Support material**	**# beams**	**q (N/m)**	**Cost ($)**
0.3183	0.012	1.83	0.494	0.3952	A514 Steel	Concrete	1	33e+4	6351.51
0.2514	0.012	1.97	0.469	0.3523	A514 Steel	Concrete	1	33e+4	6351.51

One of the design keeps only two variables ('b' and 't') changes values from the optimal design, where as the other design changes all design variables to varying degrees. The first design (first row on table above) reflects a wider column and a shallower beam height. The second row indicates narrower beam with increased height.

(c) In this case, the Normal Boundary Intersection (NBI) method is used as the multi-objective optimization algorithm. Since we are keeping the discrete variables constant and varying only the continuous variables, we can use gradient-based methods for optimization.

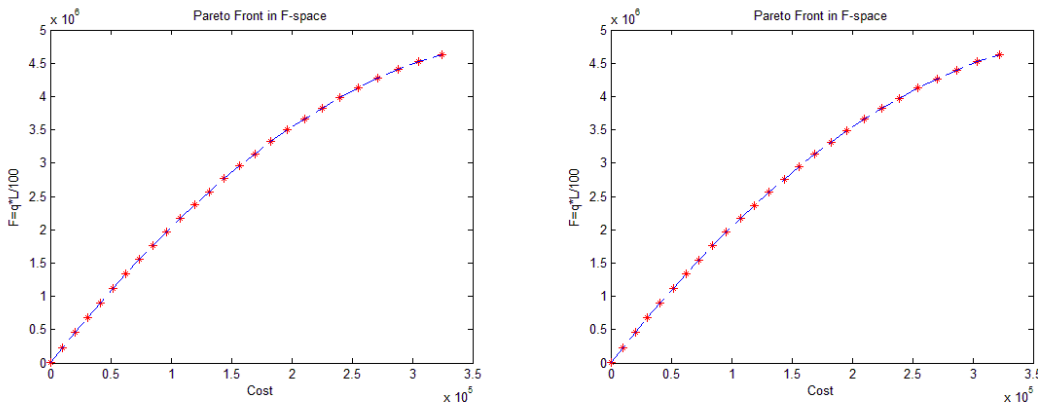


Fig. 1: (a) Pareto front without shear constraints; (b) with shear and constraint on column depth.

The active constraints are primarily the bending stress in beam and column allowable load. In the case of constraint linking the beam width and column depth, this constraint is also active. Notice that the Pareto front has two different regimes. It is linear at lower leads and is nonlinear beyond an applied load of about 2.17e+8 (N).

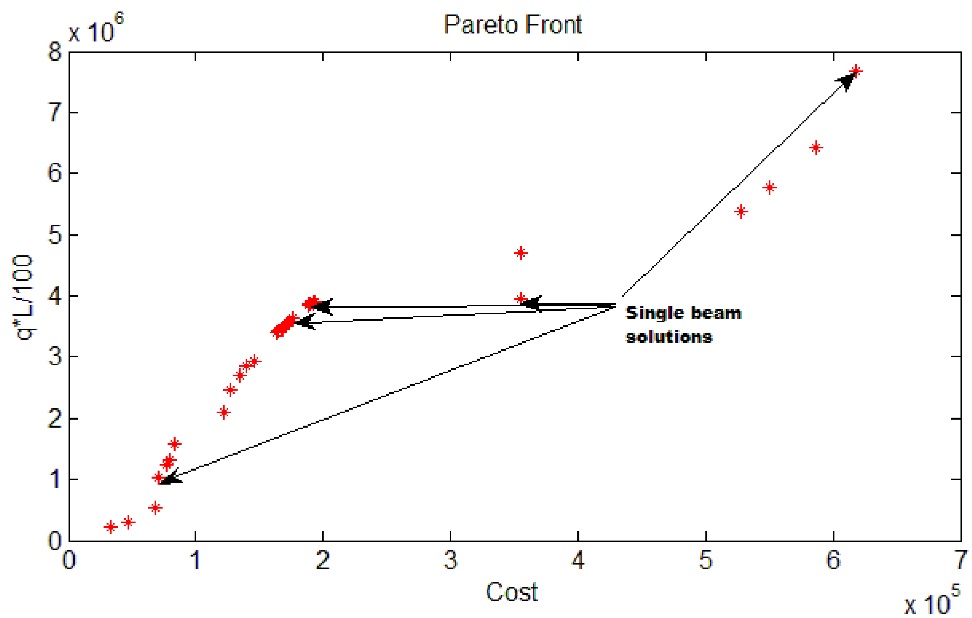
In both cases, the beam width (b) reaches its upper limit in all cases except at minimal cost design. The same is true for beam height (h). The column width (w) increases steadily as the load goes up, then alternates and finally hits its upper limit at maximal load design. The same is true for column depth (d). The beam thickness increase steadily as the applied load increase, but never hits its upper limit. There is no feasible design with given materials and number of beams for higher applied loads.

(d) The Pareto front found above is almost linear in the moderate to high load range and exhibits non-linear behavior only for extremely high loads.

In moderate to high applied load range, the plot shown above has a slope ~20.3 (N/$). This means that the cost vs. applied load (F = q*L) has a slope of ~2030 (N/$). This means that a nominal increase in cost can help support a much larger increase in applied load.

(e) The Pareto front in chosen representation is concave. This is because one of the objectives here is being maximized. If the plot is generated where cost and (–applied load) are minimized, it would be convex.

Bonus: In this case, the design space is expanded to include three additional integer design variables. This necessitates the use of heuristic multi-objective optimization methods that can handle continuous and integer variables simultaneously. A multi-objective genetic algorithm is used to generate the Pareto front in this case. The Pareto front generated for the case with constraint linking beam width to column depth, is shown below:



One can observe that it is possible to handle increased applied loads with increased number of beams with different combination of materials. There are different regimes in the Pareto front and one striking thing is that there are no solutions with the same material and number of beams combination as found in the minimal cost design before. It means that both objectives, higher load carrying capacity and lower cost designs can be met by changing the materials and number of beams. The minimal cost design found earlier for a given load becomes a dominated design now.
Another interesting feature is that the designs with 1 beam spans the different regions in the Pareto front by varying the materials used for beam and support, in addition to their cross-sectional dimensions. This illustrates the importance of material selection in design. The active constraints involve the bending stress in beam, the allowable load on the column and constraint linking column depth to beam width. In some cases (especially with higher critical loads), the critical load on column also becomes active.

The Pareto front will definitely depend on the problem formulation itself. The final front also depends on the discretization of the design variables (i.e., number of bits required to represent the design variable), the crossover and mutation rates used.