Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72, ESD.721

RPRA 2. Elements of Probability Theory

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Probability: Axiomatic Formulation

The probability of an event A is a number that satisfies the following axioms (Kolmogorov):

$$0 \leq P(A) \leq 1$$

$$P(\text{certain event}) = 1$$

For two mutually exclusive events A and B:

$$P(A \text{ or } B) = P(A) + P(B)$$

Relative-frequency interpretation

- Imagine a large number n of repetitions of the "experiment" of which A is a possible outcome.

- If A occurs k times, then its relative frequency is: $\frac{k}{n}$

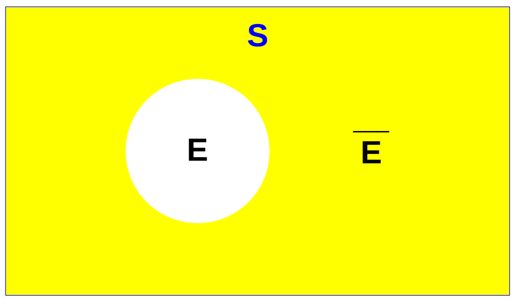
- It is postulated that: $$\lim_{n \to \infty} \frac{k}{n} \equiv P(A)$$

Degree-of-belief (Bayesian) interpretation

- **No need for “identical” trials.**
- **The concept of “likelihood” is primitive, i.e., it is meaningful to compare the likelihood of two events.**
- **P(A) < P(B) simply means that the assessor judges B to be more likely than A.**
- **Subjective probabilities must be coherent, i.e., must satisfy the mathematical theory of probability and must be consistent with the assessor’s knowledge and beliefs.**

Basic rules of probability: Negation

Venn Diagram



$$E \cup \overline{E} = S$$

$$P(E) + P(\overline{E}) = P(S) = 1 \quad \Rightarrow$$

$$P(\overline{E}) = 1 - P(E)$$

Basic rules of probability: Union

$$P\left(\bigcup_{1}^{N} A_i\right) = \sum_{i=1}^{N} P(A_i) - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} P(A_i A_j) + \ldots + (-1)^{N+1} P\left(\bigcap_{1}^{N} A_i\right)$$

Rare-Event Approximation:

$$P\left(\bigcup_{1}^{N} A_i\right) \cong \sum_{i=1}^{N} P(A_i)$$

Union (cont'd)

- **For two events:** $P(A \cup B) = P(A) + P(B) - P(AB)$

- **For mutually exclusive events:**

$$P(A \cup B) = P(A) + P(B)$$

Example: Fair Die

Sample Space: {1, 2, 3, 4, 5, 6} (discrete)

"Fair": The outcomes are equally likely (1/6).

P(even) = P(2 $\cup$ 4 $\cup$ 6) = ½ (mutually exclusive)

Union of minimal cut sets

From RPRA 1, slide 15

$$X_T = \sum_{i=1}^{N} M_i - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} M_i M_j + \ldots + (-1)^{N+1} \prod_{i=1}^{N} M_i$$

$$P(X_T) = \sum_{i=1}^{N} P(M_i) - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} P(M_i M_j) + \ldots + (-1)^{N+1} P\left(\prod_{i=1}^{N} M_i \right)$$

Rare-event approximation: $P(X_T) \cong \sum_{1}^{N} P(M_i)$

Upper and lower bounds

$$P(X_T) = \sum_{i=1}^{N} P(M_i) - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} P(M_i M_j) + \ldots + (-1)^{N+1} \prod_{i=1}^{N} P(M_i)$$

$$P(X_T) \le \sum_{1}^{N} P(M_i)$$

The first "term," i.e., sum, gives an upper bound.

$$P(X_T) \ge \sum_{i=1}^{N} P(M_i) - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} P(M_i M_j)$$

The first two "terms," give a lower bound.

Conditional probability

$$P(A/B) \equiv \frac{P(AB)}{P(B)}$$

$$P(AB) = P(A/B)P(B) = P(B/A)P(A)$$

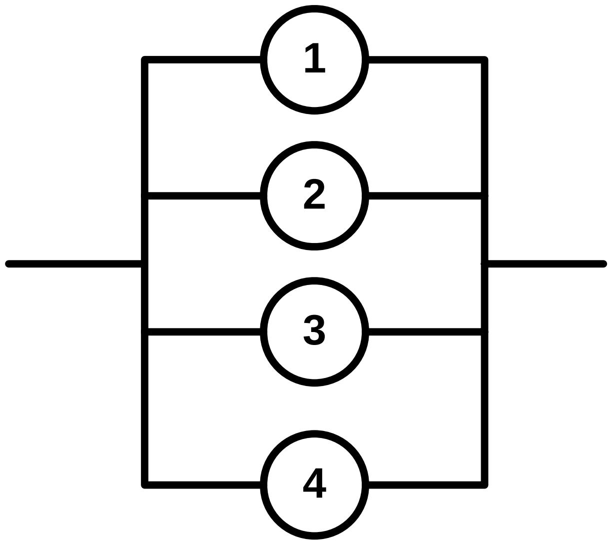
For independent events:

$$P(AB) = P(A)P(B)$$

$$P(B/A) = P(B)$$

Learning that A is true has no impact on our probability of B.

Example: 2-out-of-4 System



$M_1 = X_1 X_2 X_3$ $\qquad M_2 = X_2 X_3 X_4$

$M_3 = X_3 X_4 X_1$ $\qquad M_4 = X_1 X_2 X_4$

$$X_T = 1 - (1 - M_1)(1 - M_2)(1 - M_3)(1 - M_4)$$

$$X_T = (X_1 X_2 X_3 + X_2 X_3 X_4 + X_3 X_4 X_1 + X_1 X_2 X_4) - 3X_1 X_2 X_3 X_4$$

2-out-of-4 System (cont'd)

$$P(X_T = 1) = P(X_1 X_2 X_3 + X_2 X_3 X_4 + X_3 X_4 X_1 + X_1 X_2 X_4) - 3P(X_1 X_2 X_3 X_4)$$

Assume that the components are independent and nominally identical with failure probability q. Then,

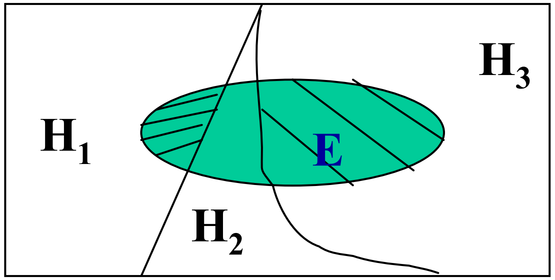
$$P(X_T = 1) = 4q^3 - 3q^4$$

Rare-event approximation: $P(X_T = 1) \cong 4q^3$

Updating probabilities (1)

- **The events, H_i, i = 1…N, are mutually exclusive and exhaustive, i.e., $H_i \cap H_j = \emptyset$, for $i \neq j$, $\cup H_i = S$, the sample space.**

- **Their probabilities are $P(H_i)$.**



- **Given an event E, we can always write**

$$P(E) = \sum_{1}^{N} P(E/H_i)P(H_i)$$

Updating probabilities (2)

- Evidence E becomes available.

- What are the new (updated) probabilities $P(H_i/E)$?
  Start with the definition of conditional probabilities, slide 11.

$$P(EH_i) = P(E/H_i)P(H_i) = P(H_i/E)P(E) \quad \Rightarrow$$

$$P(H_i/E) = \frac{P(E/H_i)P(H_i)}{P(E)}$$

- Using the expression on slide 14 for P(E), we get

Bayes' Theorem

Likelihood of the Evidence

Prior Probability

$$P(H_i/E) = \frac{P(E/H_i)P(H_i)}{\sum_{1}^{N} P(E/H_i)P(H_i)}$$

Posterior Probability

Example: Let's Make A Deal

- Suppose that you are on a TV game show and the host has offered you what's behind any one of three doors. You are told that behind one of the doors is a Ferrari, but behind each of the other two doors is a Yugo. You select door A.

  At this time, the host opens up door B and reveals a Yugo. He offers you a deal. You can keep door A or you can trade it for door C.

- What do you do?

Let's Make A Deal: Solution (1)

- **Setting up the problem in mathematical terms:**
  - ***A = {The Ferrari is behind Door A}***
  - ***B = {The Ferrari is behind Door B}***
  - ***C = {The Ferrari is behind Door C}***
- **The events A, B, C are mutually exclusive and exhaustive.**
- **P(A) = P(B) = P(C) = 1/3**

E = {The host opens door B and a Yugo is behind it}

What is P(A/E)? ⇒ Bayes' Theorem

Let's Make A Deal: Solution (2)

$$P(A/E) =$$

$$= \frac{P(E/A)P(A)}{P(E/A)P(A) + P(E/B)P(B) + P(E/C)P(C)}$$

But

P(E/B) = 0 **(A Yugo is behind door B).**

P(E/C) = 1 **(The host must open door B, if the Ferrari is behind door C; he cannot open door A under any circumstances).**

Let's Make A Deal: Solution (3)

- Let $P(A/E) = x$ and $P(E/A) = p$
- Bayes' theorem gives:

$$x = \frac{p}{1+p}$$

Therefore

- For $P(E/A) = p = 1/2$ (the host opens door B randomly, if the Ferrari is behind door A)

$\Rightarrow$ $P(A/E) = x = 1/3 = P(A)$ (the evidence has had no impact)

Let's Make A Deal: Solution (4)

- **Since $P(A/E) + P(C/E) = 1 \Rightarrow$**
- **$P(C/E) = 1 - P(A/E) = 2/3 \Rightarrow$**

$\Rightarrow$ ***The player should switch to door C***

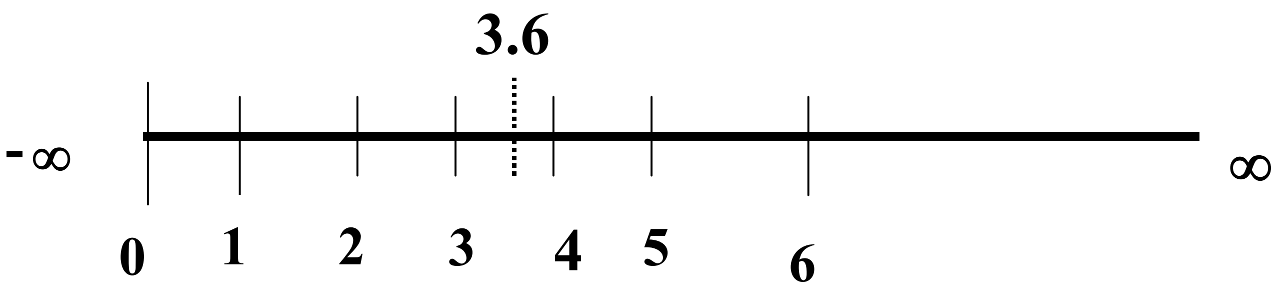
- **For $P(E/A) = p = 1$ (the host always opens door B, if the Ferrari is behind door A)**

$\Rightarrow$ **$P(A/E) = 1/2 \Rightarrow P(C/E) = 1/2$,** ***switching to door C does not offer any advantage.***

Random Variables

- ***Sample Space:*** **The set of all possible outcomes of an experiment.**
- ***Random Variable:*** **A function that maps sample points onto the real line.**
- **Example: For a die $\Rightarrow$ S = {1,2,3,4,5,6}**
- **For the coin: S = {H, T} ≡ {0, 1}**

Events



We say that $\{X \leq x\}$ is an event, where x is *any number* on the real line.

For example (die experiment):

$\{X \leq 3.6\} = \{1, 2, 3\} \equiv \{1 \text{ or } 2 \text{ or } 3\}$

$\{X \leq 96\} = S$ (the certain event)

$\{X \leq -62\} = \varnothing$ (the impossible event)

Sample Spaces

- **The SS for the die is an example of a *discrete sample space* and X is a *discrete random variable (DRV).***

- **A SS is *discrete* if it has a finite or countably infinite number of sample points.**

- **A SS is *continuous* if it has an infinite (and uncountable) number of sample points. The corresponding RV is a *continuous random variable (CRV).***

- **<u>Example:</u>**

  $\{T \leq t\}$ = {failure occurs before t}

Cumulative Distribution Function (CDF)

- **The *cumulative distribution function (CDF)* is**

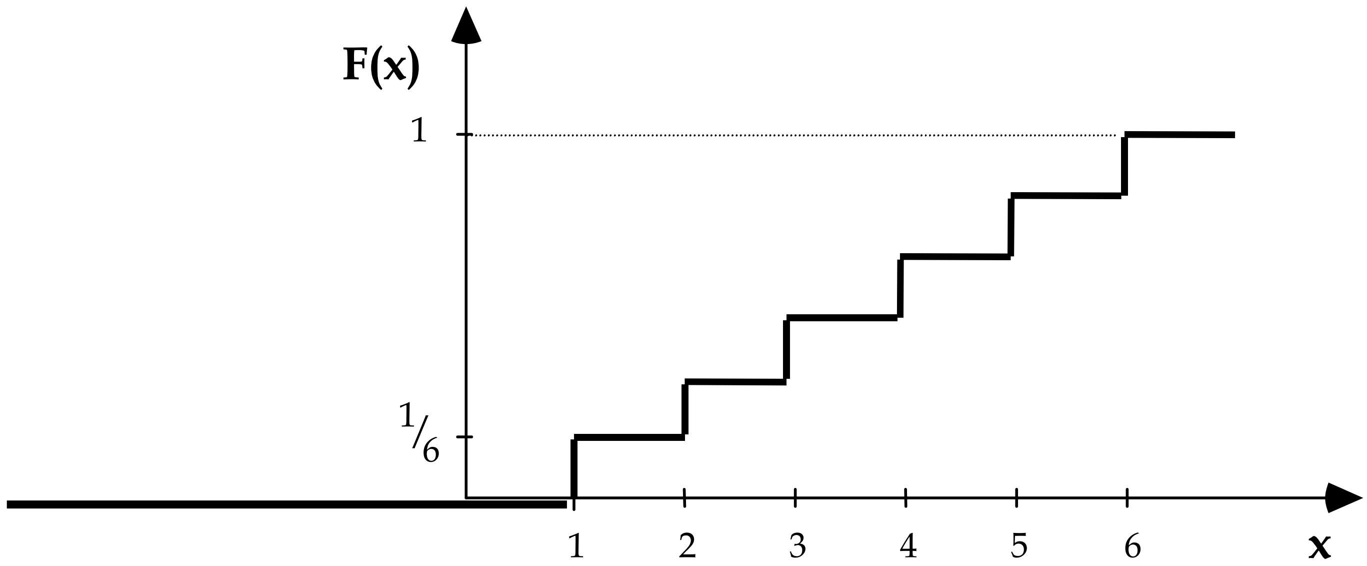
  $$F(x) \equiv \Pr[X \le x]$$

- **This is true for both DRV and CRV.**

Properties:

1. **F(x) is a non-decreasing function of x.**
2. $F(-\infty) = 0$
3. $F(\infty) = 1$

CDF for the Die Experiment



Probability Mass Function (pmf)

- **For DRV:** ***probability mass function***

$$P(X = x_i) \equiv p_i$$

$$F(x) = \sum p_i, \quad \text{for all} \quad x_i \le x$$

$$P(S) = \sum_i p_i = 1$$ **normalization**

Example: For the die, $p_i = 1/6$ **and** $\sum_1^6 p_i = 1$

$$F(2.3) = P(1 \cup 2) = \sum_1^2 p_i = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Probability Density Function (pdf)

$$f(x)dx = P\{x < X < x+dx\}$$

$$f(x) = \frac{dF(x)}{dx} \qquad F(x) = \int_{-\infty}^{x} f(s)ds$$

$$P(S) = F(\infty) = \int_{-\infty}^{\infty} f(s)ds = 1$$

normalization

Example of a pdf (1)

- Determine k so that

$$f(x) = kx^2, \qquad \text{for} \qquad 0 \le x \le 1$$

$$f(x) = 0, \qquad \text{otherwise}$$

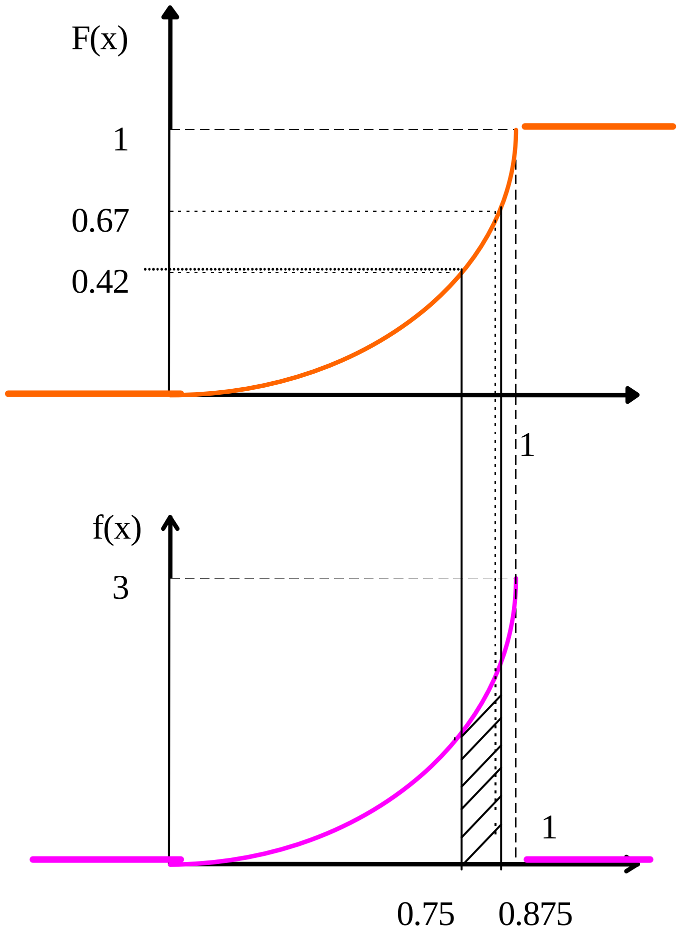
is a pdf.

Answer:

The normalization condition gives:

$$\int_0^1 kx^2 dx = 1 \qquad \Rightarrow \qquad k = 3$$

Example of a pdf (2)



$$F(x) = x^3$$

$$F(0.875) - F(0.75) = \int_{0.75}^{0.875} 3x^2 dx =$$

= 0.67 - 0.42 = 0.25 =

= P{0.75 < X < 0.875}

Moments

Expected (or mean, or average) value

$$E[X] \equiv m \equiv \begin{cases} \int_{-\infty}^{\infty} x f(x) dx & \text{CRV} \\ \sum_j x_j p_j & \text{DRV} \end{cases}$$

Variance (standard deviation σ)

$$E\left[(X-m)^2\right] \equiv \sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (x-m)^2 f(x) dx & \text{CRV} \\ \sum_j (x_j - m)^2 p_j & \text{DRV} \end{cases}$$

Percentiles

- ***Median*: The value x_m for which**
- **$F(x_m) = 0.50$**
- **For CRV we define the 100γ percentile as that value of x for which**

$$\int_{-\infty}^{x_\gamma} f(x)\,dx = \gamma$$

Example

$$m = \int_0^1 3x^3 dx = 0.75$$

$$\sigma^2 = \int_0^1 3(x - 0.75)^2 x^2 dx = 0.0375 \qquad \sigma = 0.194$$

$$F(x_m) = x_m^3 = 0.5 \quad \Rightarrow \quad x_m = 0.79$$

$$x_{0.05}^3 = 0.05 \quad \Rightarrow \quad x_{0.05} = 0.37$$

$$x_{0.95}^3 = 0.95 \quad \Rightarrow \quad x_{0.95} = 0.98$$