Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72, ESD.721

DA 5. Risk Aversion

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Calibration of utility functions

- We can apply a positive linear transformation to a utility function and get an equivalent utility function.

$$\pi(x) = a\, U(x) + b \qquad a>0$$

- A calibrated utility function is such that

$$\pi(C_*) = 0 \quad \text{and } \pi(C^*) = 1$$

Example

- Suppose that it has been determined that

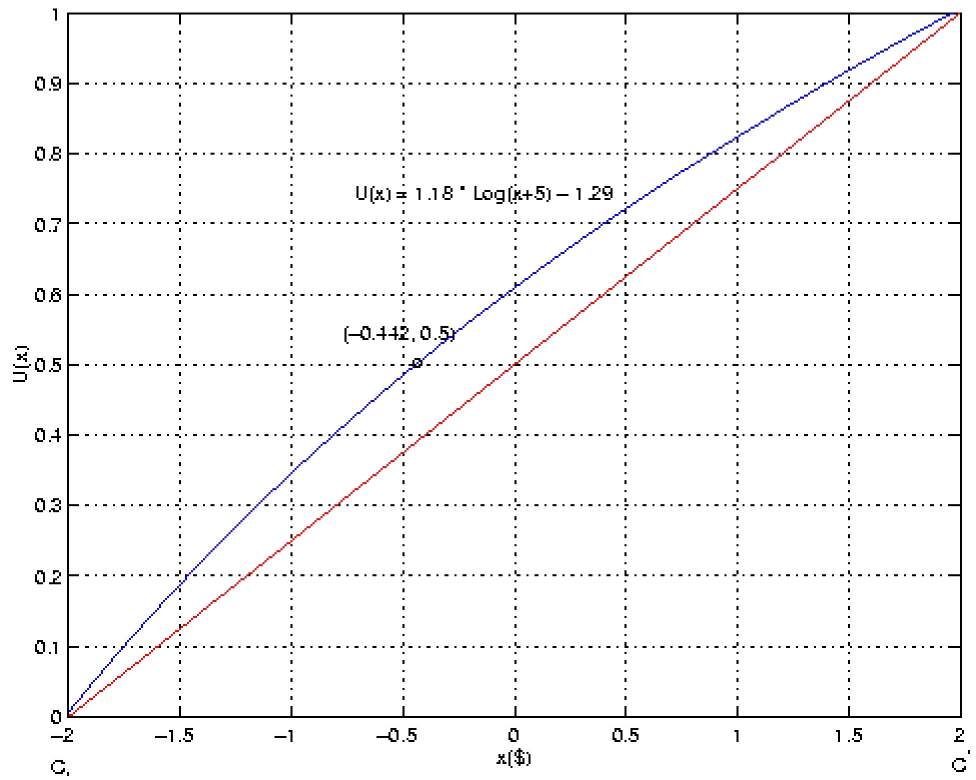
  $U(x) = \ln(x+5)$ for $-4.5 \leq x \leq 4.5$ (in \$ million)

- Let $C^* = 2$ and $C_* = -2$

- Let $\pi(x) = a\ln(x+5) + b$. Then,

  $1 = a\ln(7) + b$ and $0 = a\ln(3) + b$

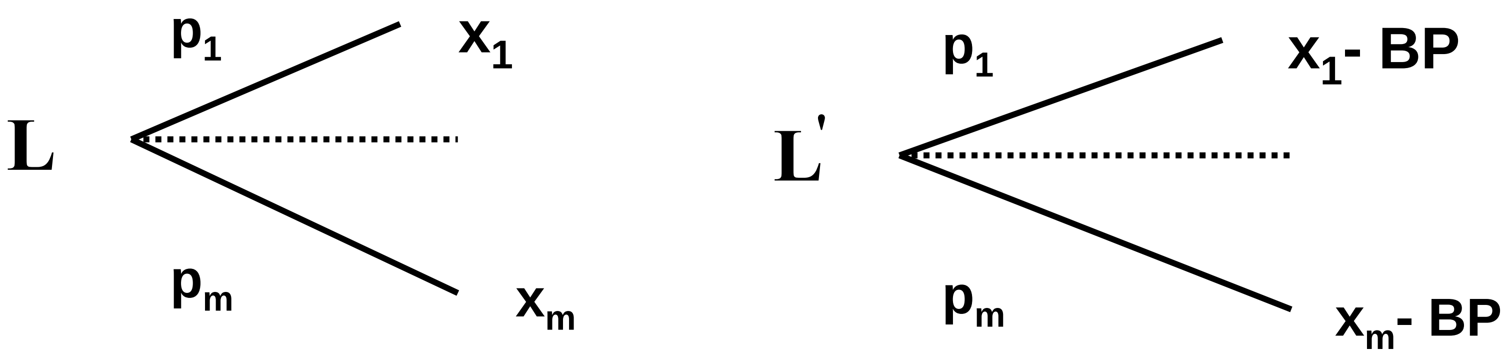
  to obtain a = 1.18 and b = -1.29


U(x) = 1.18 * Log(x+5) – 1.29
(–0.442, 0.5)
U(x)
x($)
C*
C*
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
–2
–1.5
–1
–0.5
0
0.5
1
1.5
2

The Buying Price for a Lottery

- **It is the purchase price at which the DM is indifferent between the alternatives of buying the lottery and not buying it.**

- **Let π(x) the DM's utility function with π(0) the utility of his present assets, i.e.,** ***before*** **he buys the lottery.**



$$\pi(0) = U(L') = \sum_i p_i \pi(x_i - BP)$$

Example

- $\pi(x) = 1.18 \ln(x+5) - 1.29 \quad \Rightarrow \quad \pi(0) = 0.61$
- $L(1, 0; 0.5, 0.5)$
- $0.5\, \pi(1-BP) + 0.5\, \pi(0-BP) = 0.61 \quad \Rightarrow$
- $1.18 \ln(6-BP) - 1.29 + 1.18 \ln(5-BP) - 1.29 = 1.22$
- $\ln(6-BP) + \ln(5-BP) = 3.22 \quad \Rightarrow$

Example (cont'd)

- **$\ln[(6\text{-}BP)\,(5\text{-}BP)] = 3.22$**
- **$(BP)^2 - 11(BP) + 30 = \exp(3.22) = 25.04$**
- **$(BP)^2 - 11(BP) + 4.96 = 0$**
- **$BP = 0.47$ (the other root is 10.53 and is rejected)**
- **$EMV = 1x0.5 + 0x0.5 = 0.5 > 0.47 \Rightarrow$ risk aversion**

The Selling Price of a Lottery

- **SP = CE (certainty equivalent)**
- **We now interpret x = 0 to represent the DM's total assets *except* of the lottery.**
- **Utility of present wealth (situation) is $\pi(L)$.**
- **$\pi(x) = 1.18 \ln(x+5) - 1.29 \quad \Rightarrow \quad \pi(0) = 0.6091$**

Example

- $\pi(x) = 1.18 \ln(x+5) - 1.29 \quad \Rightarrow \quad \pi(0) = 0.6091$
- L(1, 0; 0.5, 0.5)
- $\pi(L) = 0.5\,\pi(1) + 0.5\,\pi(0) = 0.5 \times 0.8243 + 0.5 \times 0.6091 \quad \Rightarrow$
- $\pi(L) = 0.7167 = \pi(CE) \quad \Rightarrow$ Using the figure on p. 4 we get
- $CE \cong 0.4771 < EMV = 0.5$

Risk Aversion

- **The DM is always willing to sell any lottery for less than its expected monetary value.**
- $L(x_1, x_2; p_1, p_2)$ $\qquad$ $EMV = p_1 x_1 + p_2 x_2$
- $U(p_1 x_1 + p_2 x_2) = U(EMV) > p_1 U(x_1) + p_2 U(x_2)$
- **A DM with a *concave* utility for money will refuse a monetarily fair bet and is said to be *risk averse*.**

Example

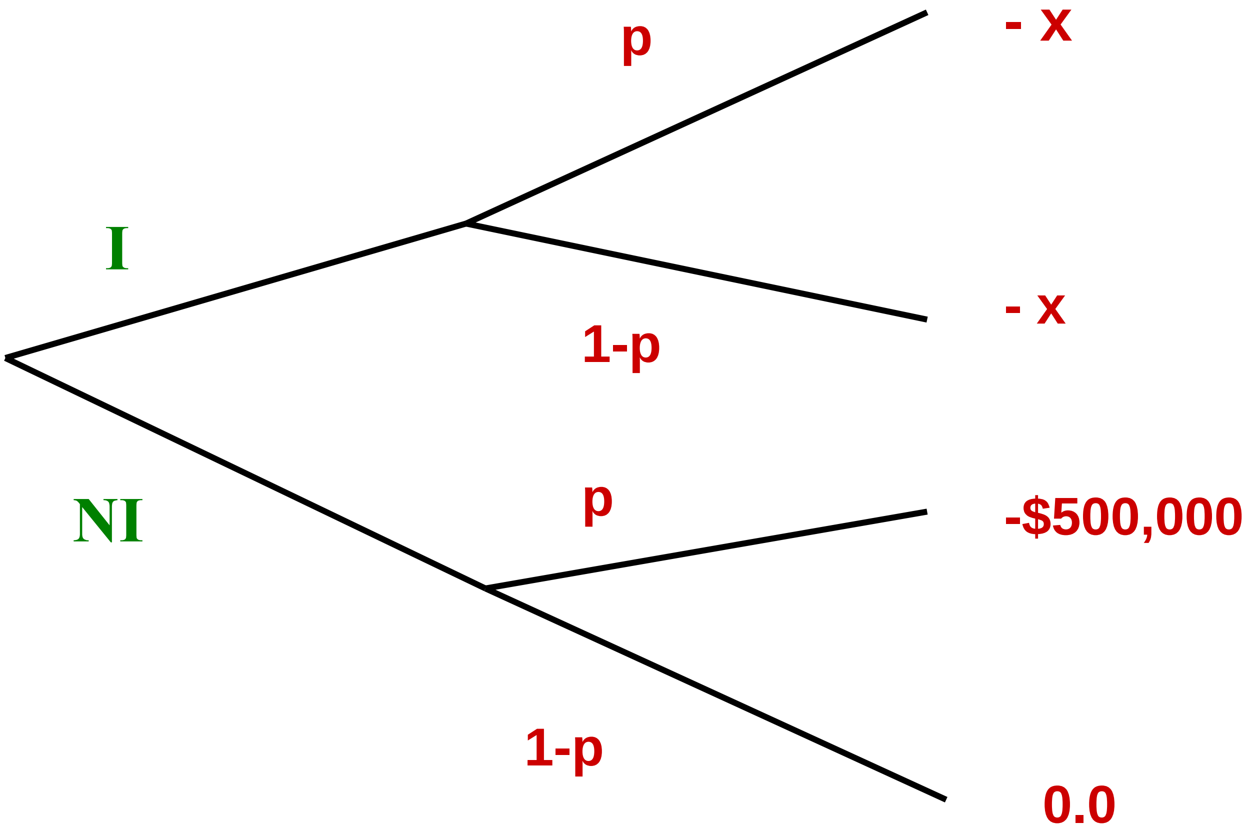
- **L(1, 0; 0.5, 0.5)**
- **EMV = \$0.5 M ⇒ $\pi(0.5) = 0.6091$**
- **$0.5\,\pi(1) + 0.5\,\pi(0) = 0.7167$ ⇒ SP = \$0.4771**
- ***Risk Premium* ≡ EMV – CE = 0.5 – 0.4771 ⇒**
- **RP = \$0.0229 M**

Risk Aversion and Insurance

- You own a house worth $500,000.
- A fire ($p = 10^{-2}$ per year) may destroy it completely.
- Your utility function is

- $\pi(x) = 1.18\ln(x+5) - 1.29$; $C^* = \$2M$, $C_* = -\$2M$
- How much premium would you be willing to pay?

Decision Tree



Utilities

- $\pi(I) = \pi(-x)$ $\qquad$ $\pi(NI) = p\,\pi(-0.5) + (1-p)\,\pi(0.0)$
- $\pi(-0.5) = 1.18\ln(4.5) - 1.29 = 0.4848$
- $\pi(0.0) = 0.6091$
- $\pi(NI) = 0.4848 \times 10^{-2} + (1-10^{-2}) \times 0.6091 = 0.6078$
- $\pi(-x) = 0.6078 \Rightarrow x = \$5{,}661$

Your Perspective

- You are willing to pay up to $5,661 to insure your house.
- The expected loss is

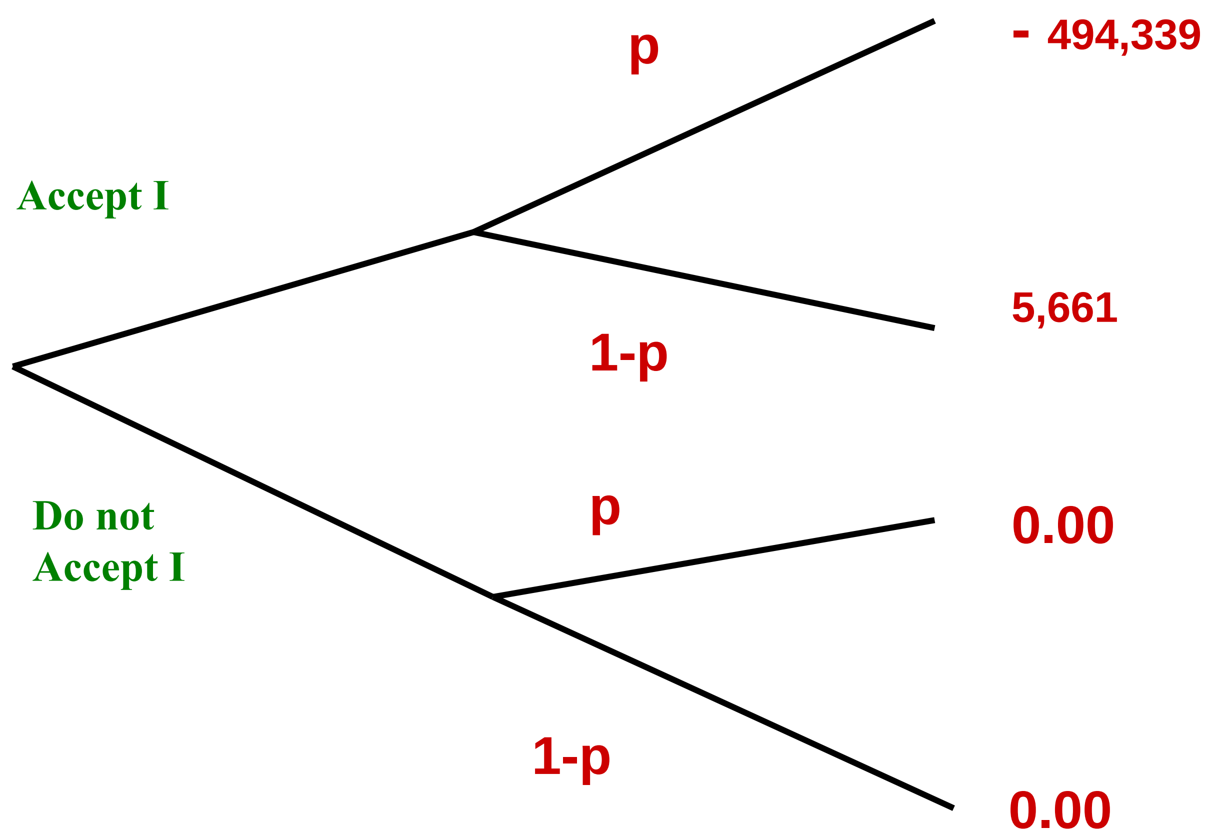
10^{-2} x $500,000 = $5,000 < $5,661 $\Rightarrow$

You are willing to pay more than the expected loss because you are risk averse.

The Company's Perspective

- Why would the insurance company agree to insure you?
- The company may win $5,661 with probability 0.99 or lose $494,339 with probability 10^{-2}.

The Company's Decision Tree



The Company's Perspective (2)

- The company has $1 billion dollars in assets. Its wealth is either $999,505,661 with probability 10^{-2}, or $1,000,005,661 with probability 0.99.
- For such small changes, the company's utility of money is linear, i.e., the company makes decisions using the EMV.
- EMV = 5,661 x 0.99 - 494,339 x 10^{-2} = $661

The Company's Perspective (3)

- The alternative is for the company to refuse the premium, in which case the EMV is zero.

- The company should agree to insure the house.

- <u>Note:</u> The company's overhead expenses have not been factored in.

Assessment Using Certainty Equivalents

1. Set $U(C_*) = 0$ and $U(C^*) = 1$

2. Consider the reference lottery

   $L_1(C^*, C_*; 0.5, 0.5)$

- Its certainty equivalent is derived by solving

$$U(CE_1) = 0.5\, U(C^*) + 0.5\, U(C_*) = 0.5$$

Assessment Using Certainty Equivalents (2)

Thus, we have found a third point of the utility function, that with utility 0.5.

3. Repeat the process with a new reference lottery

$$L_2(C^*, CE_1; 0.5, 0.5)$$

Solve

$$U(CE_2) = 0.5\ U(C^*) + 0.5\ U(CE_1) = 0.75$$

to get a fourth point of the utility function, CE_2.

Assessment Using Certainty Equivalents (3)

4. Repeat using $L_3(CE_1, C_*; 0.5, 0.5) \Rightarrow$

$$U(CE_3) = 0.5\ U(CE_1) + 0.5\ U(C_*) = 0.25$$

to get a fifth point, and so on.