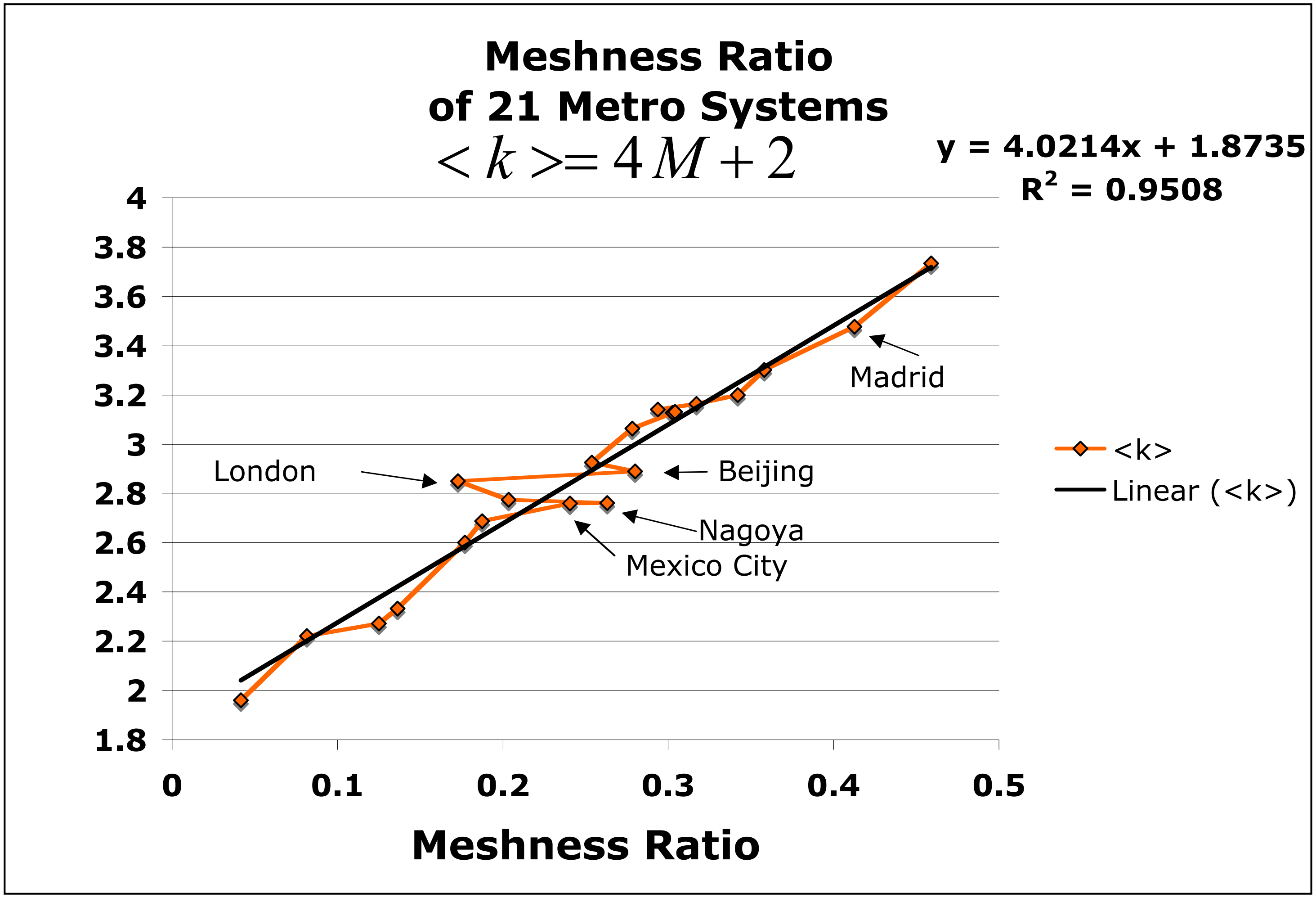
Basic Network Metrics and Operations

- Meshness ratio
- Degree correlation
  - Joint degree distribution
  - K-nearest neighbors
  - Pearson degree correlation
- Rich club metric
- Degree-preserving rewiring
- Generating a graph that has a specified degree sequence
- Finding Pearson degree correlation
- Finding communities

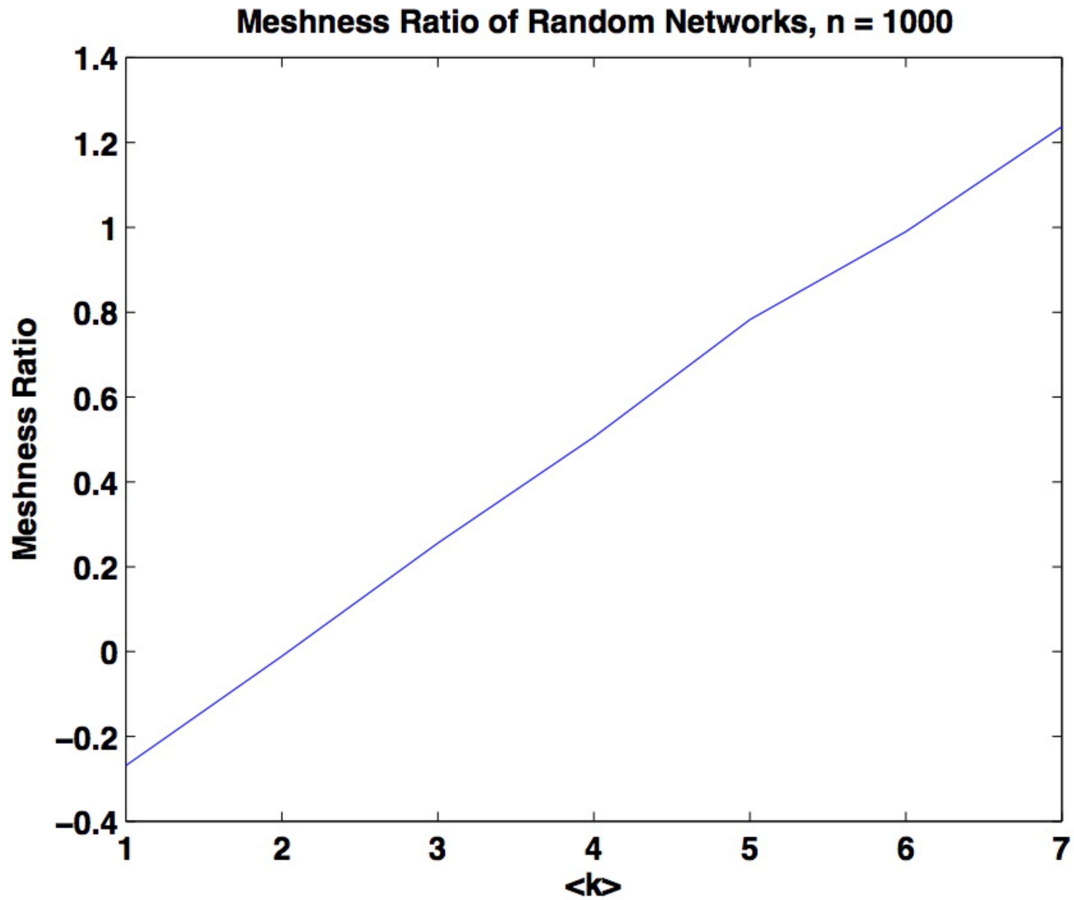
Meshness Ratio

- Exploits Euler's formula for planar graphs
- Is applied to non-planar graphs as well, not used enough for a basis for comparison to have built up yet
- Meshness = number of closed faces = m-n+2
- Max meshness = 2n-4
- Ratio = (m-n+2)/(2n-4)
- This varies between zero and 1
- "Meshy" networks seem to have mr ~ 0.3 but these are usually almost planar, such as metro systems

Meshness Ratio of Metro Systems



Meshness of Random Networks



CAIDA Paper on Internet Structure

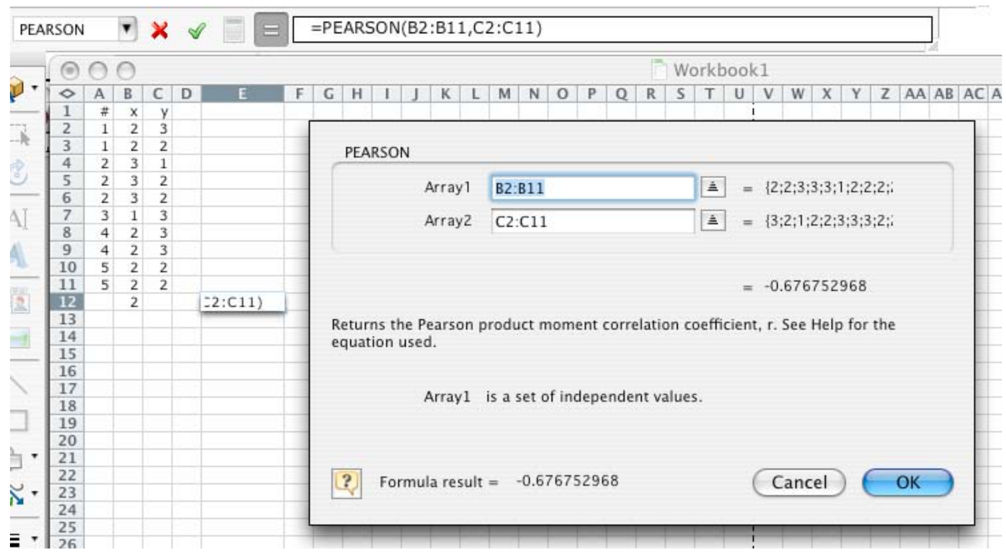
- Nice review and comparison of many metrics
- Follows up early 2000s papers purporting to find the structure of the internet
- Shows that there are three ways to do this, each approximate, using different methods, each with a bias
- Shows that each way gives different results, providing caution about artifacts inherent in data collection
- Joint Degree Distribution (JDD) seems to be the best metric

Degree Correlation *r*

- This is a subset of "homophily" meaning the extent to which nodes are alike
- Degree correlation is measured using the Pearson correlation function
- Also called "assortativity" and "disassortativity" in social network analysis
- r is positive if nodes of similar degree are linked - assortative (not the same as big to big)
- r is negative if nodes of dissimilar degree are linked - disassortative (not the same as big to small)
- Bigger magnitude of r indicates higher tendency for the specified linkage

Calculating r

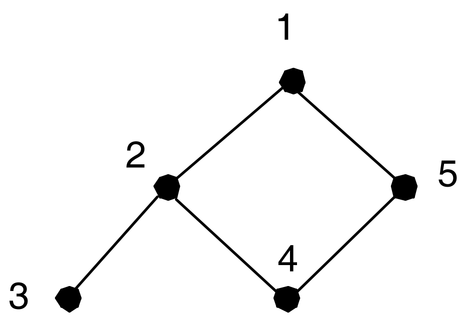
$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum \ (y-\bar{y})^2}}$$



#	x	y
1	2	3
1	2	2
2	3	1
2	3	2
2	3	2
3	1	3
4	2	3
4	2	3
5	2	2
5	2	2

$\bar{x} = 2.2$

$\bar{y} = 2.2$



r = -0.676752968 using Pearson function in Excel

Note: if all nodes have the same k then r = 0/0

Calculating x-bar

$$\bar{x} = \frac{\text{sum of column values}}{\text{number of column values}}$$

node	x	y	
1	2	3	
1	2	2	
2	3	1	
2	3	2	
2	3	2	
3	1	3	
4	2	3	
4	2	3	
5	2	2	
5	2	2	
average	2.2		pearson
			-0.676753

$$\bar{x} = 2.2$$

$$\bar{y} = 2.2$$

each node of degree k creates k rows with k in each row

number of rows $=$ sum of entries in $kvec(A) = sum(k_i)$

$$kvec(A) = 2 \;\; 3 \;\; 1 \;\; 2 \;\; 2$$

$$sum(kvec(A)) = 10$$

sum of the k row entries for each $k = k * k = k^2$

sum of all such row entries $=$ $sum(k_i^2) = 22$

$$\bar{x} = \frac{\sum_i k_i^2}{\sum_i k_i} = \frac{\frac{1}{n}\sum_{i=1}^{n} k_i^2}{\frac{1}{n}\sum_{i=1}^{n} k_i} = \frac{< k^2 >}{< k >} = 2.2$$

$\bar{x} \geq \frac{< k >^2}{< k >} = < k >$ so $\bar{x}$ is a measure of the variation in k

Matlab for Pearson (symmetric)

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2 \sum (y-\bar{y})^2}}$$

Look at numerator, ignore xbar for the moment

$$\sum \left(x_i y_j\right) = x_i^{'} \delta_{ij} y_j = x^{'} A x$$

$$\delta_{ij} = 1 \;\; if \;\; i \text{ links to } j$$

$$\delta_{ij} = 0 \;\; if \;\; i \text{ does not link to } j$$

Essentially the calculation is a quadratic form.
Pearsondir does the calculation for asymmetric networks

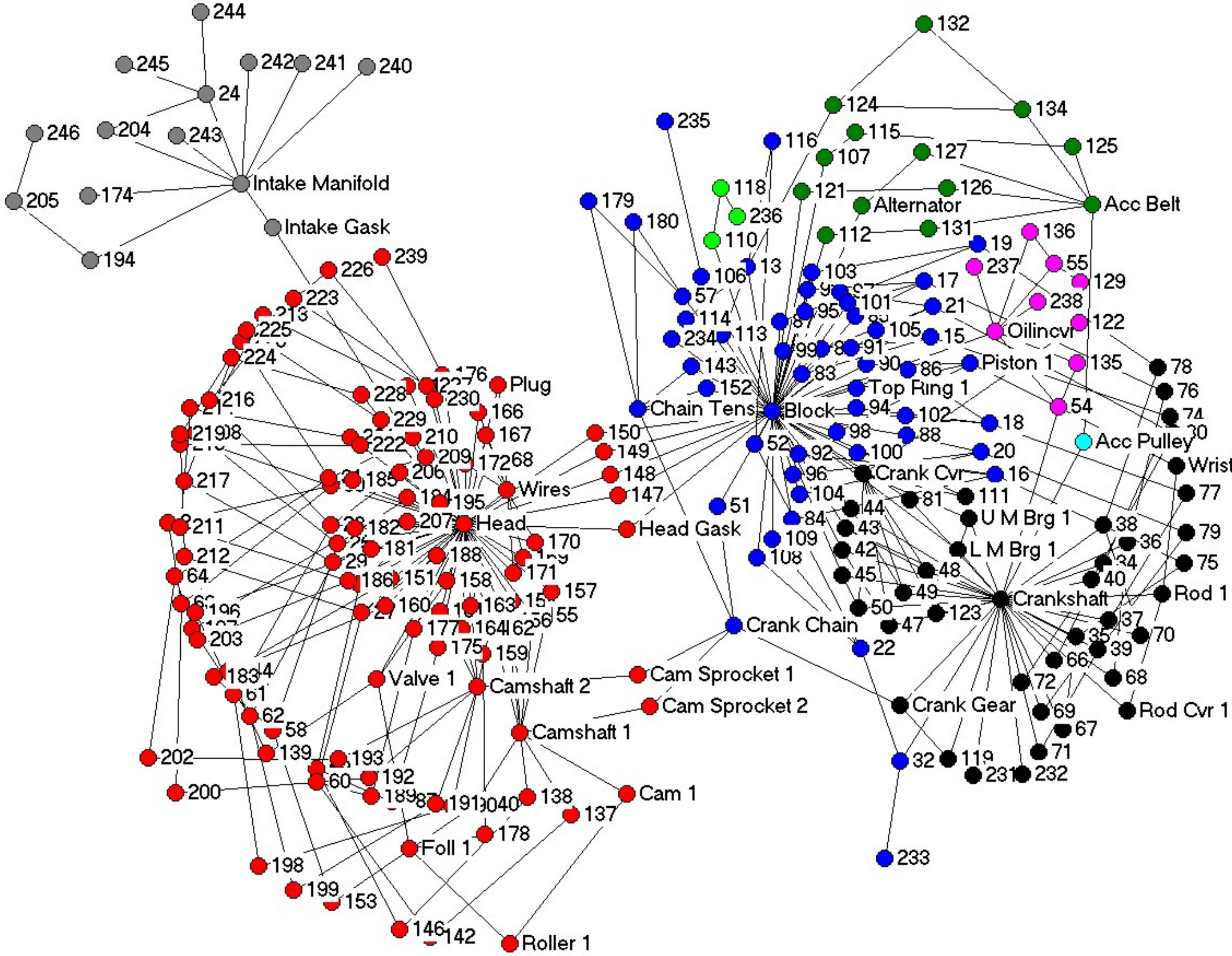
Matlab Implementation

```
function prs = pearson(A)
%calculates pearson degree correlation of A
[rows,colms]=size(A);
won=ones(rows,1);
k=won'*A;
ksum=won'*k';
ksqsum=k*k';
xbar=ksqsum/ksum;
num=(k-won'*xbar)*A*(k'-xbar*won);
kkk=(k'-xbar*won).*(k'.^.5);
denom=kkk'*kkk;
prs=num/denom;
```

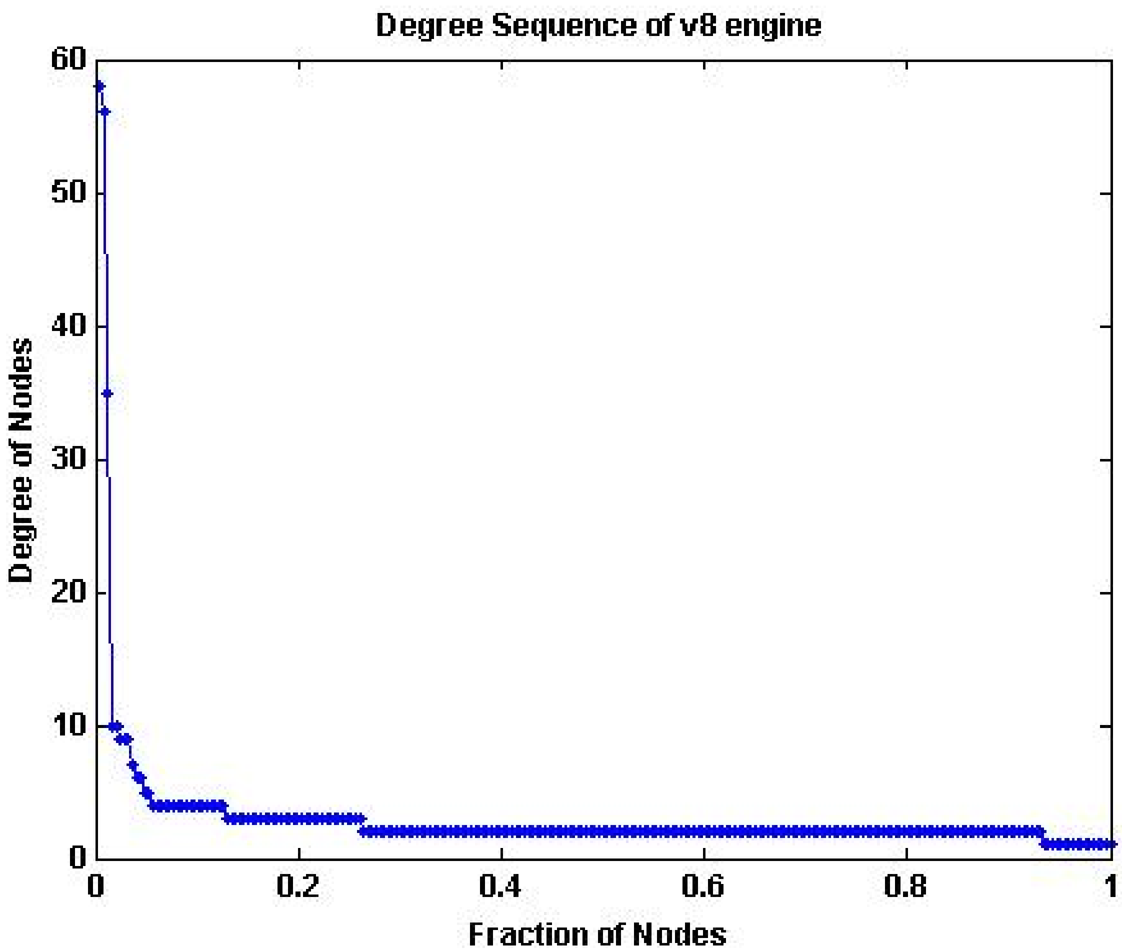

K-nearest neighbors and Joint Degree Distribution

- These seek similar info to Pearson but are more general than Pearson, which condenses all the info into a single number
- knn plots the average degree of neighbors of nodes that have degree k
  - Rising knn indicates positive degree correlation
  - Falling knn indicates negative degree correlation
- JDD1 plots cross-correlation of degree of each node with every other neighboring node
  - Shape of plot indicates sense of degree correlation

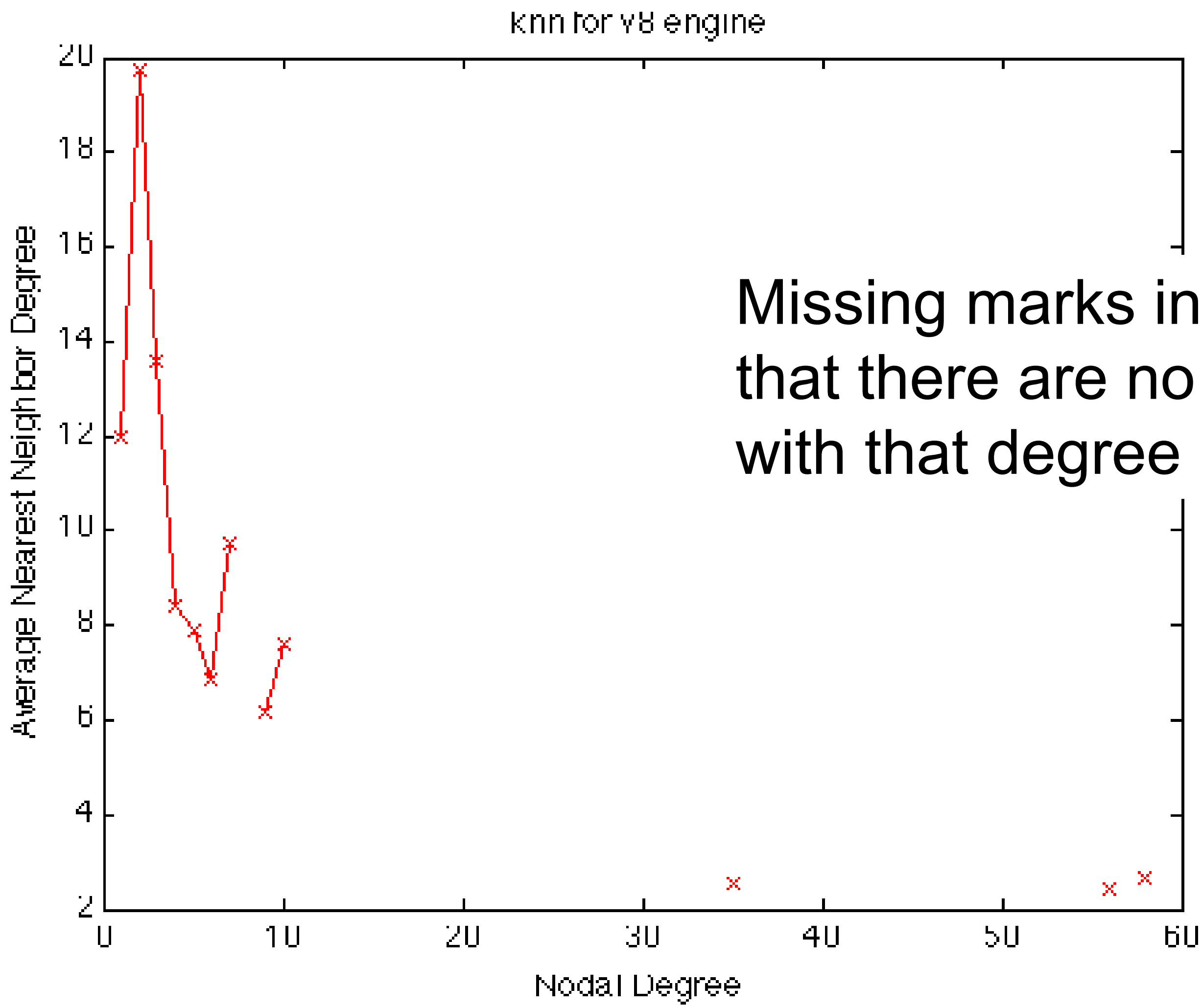
Network for V-8 Engine



Degree Distribution for V8 Engine

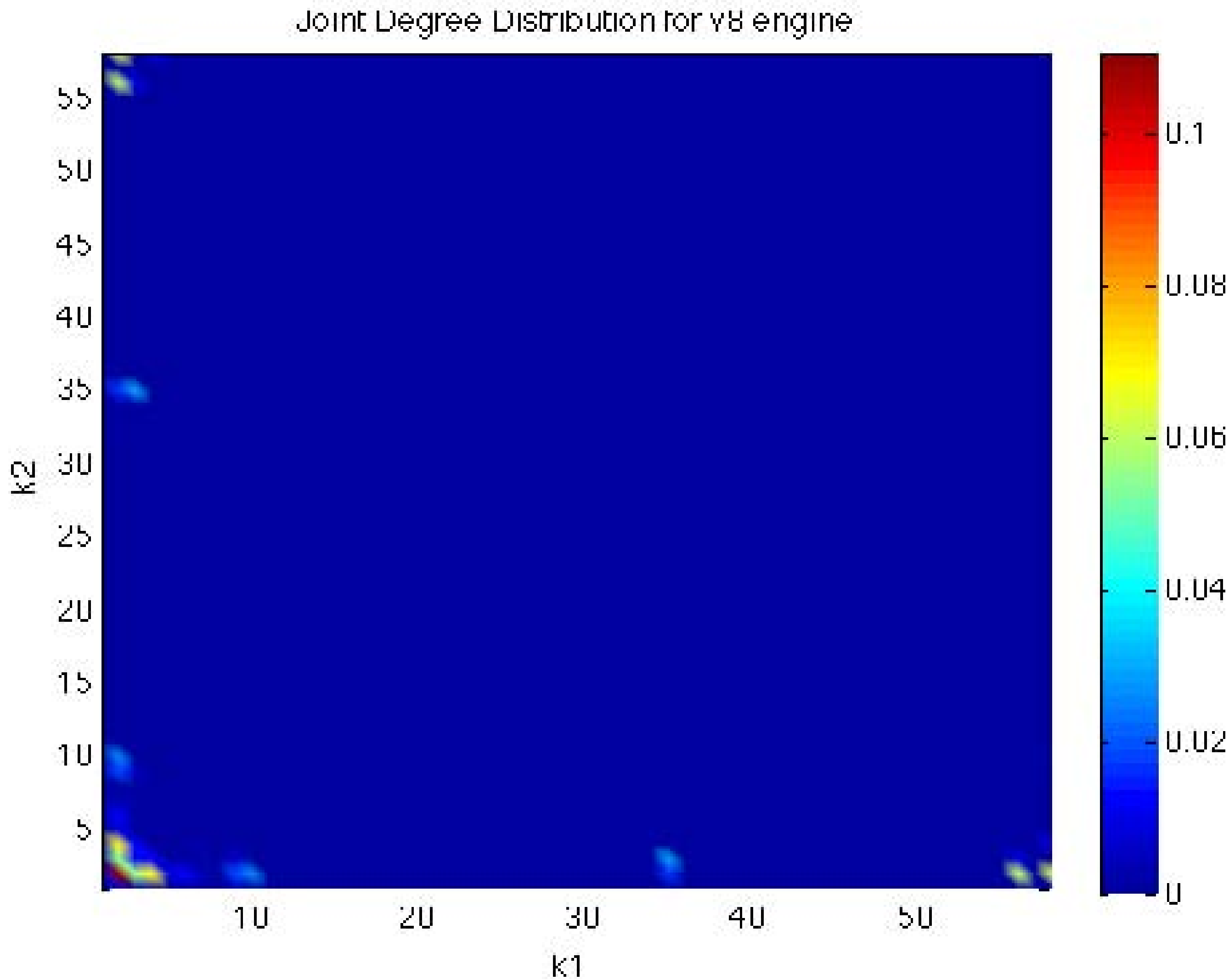


K Nearest Neighbors for V8

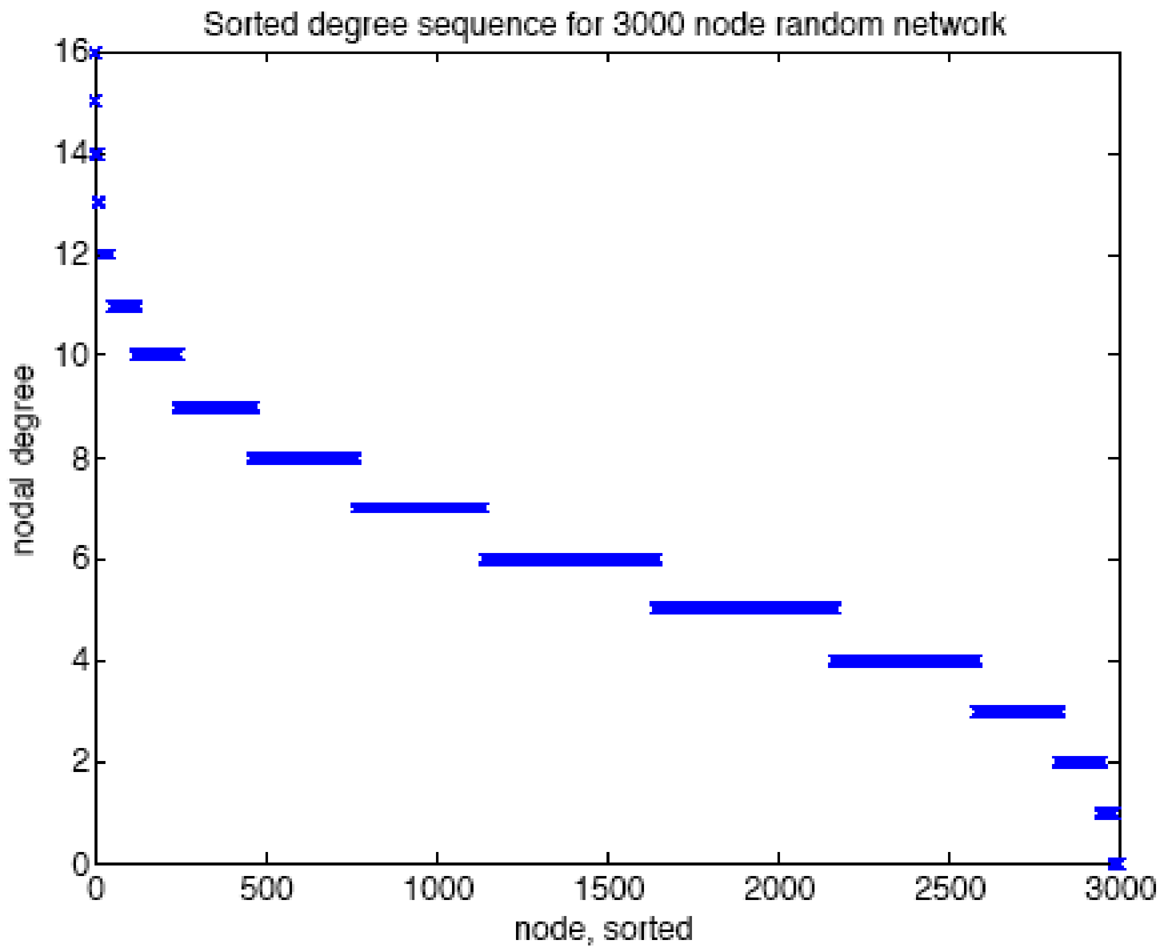


Missing marks indicate that there are no nodes with that degree

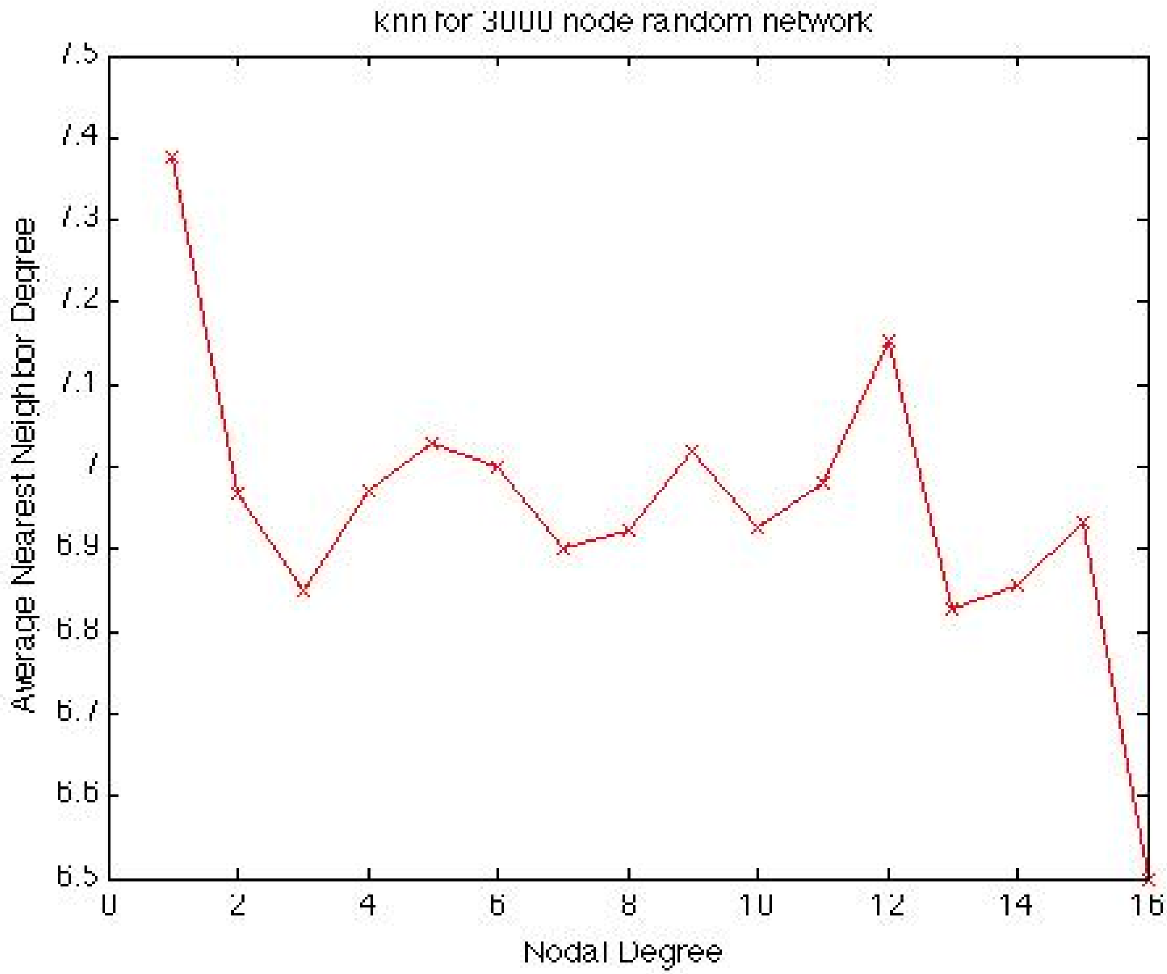
Joint Degree Distribution for V8



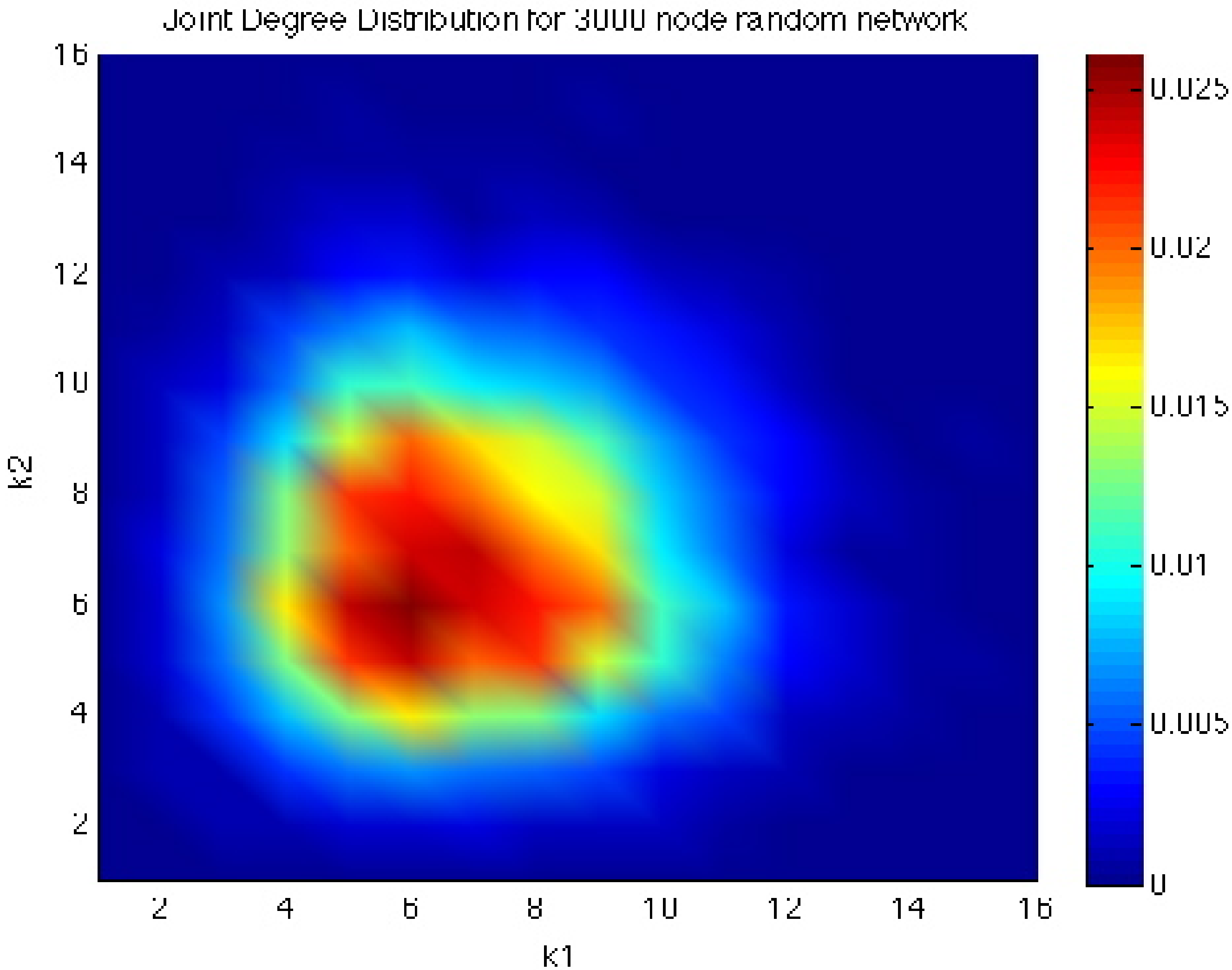
Degree Sequence of Random Network: <k> = 6



Knn for Random = $z + 1$



JDD for Random Matrix



Rewiring

- A way to deliberately transform a graph
- Several ways this is done
  - Unhooking one end of an edge and hooking it in somewhere else
  - Adding a new edge
  - Pairwise rewiring that preserves the original degree sequence
    - This can disconnect the graph unless you take care to reject rewirings that do so

Rewiring - 2

Preserving degree

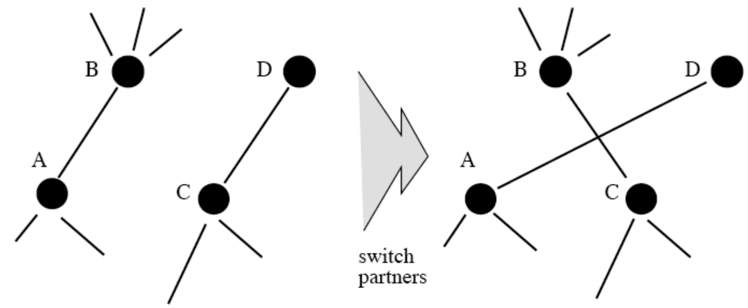


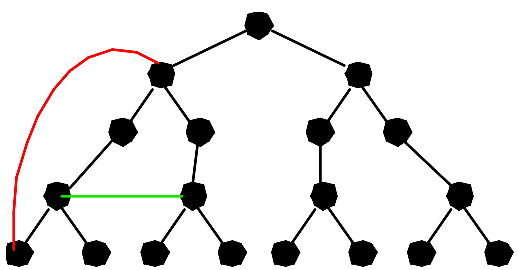
FIG. 1. One elementary step of the local rewiring algorithm. A pair of edges A—B and C—D is randomly selected. They are then rewired in such a way that A becomes connected to D, and C - to B, provided that none of these edges already exist in the network, in which case the rewiring step is aborted, and a new pair of edges is selected. The last restriction prevents the appearance of multiple edges connecting the same pair of nodes.

Detection of Topological Patterns in Complex Networks: Correlation Profile of the Internet

Sergei Maslov[1], Kim Sneppen[2,3], Alexei Zaliznyak[1]
arXiv:cond-mat/0205379 v2 6 Nov 2002

Unhook-rehook links

Add links



Information exchange and the robustness of organizational networks

Peter Sheridan Dodds*†, Duncan J. Watts*‡§, and Charles F. Sabel¶

Degree-preserving Pair-wise Rewiring

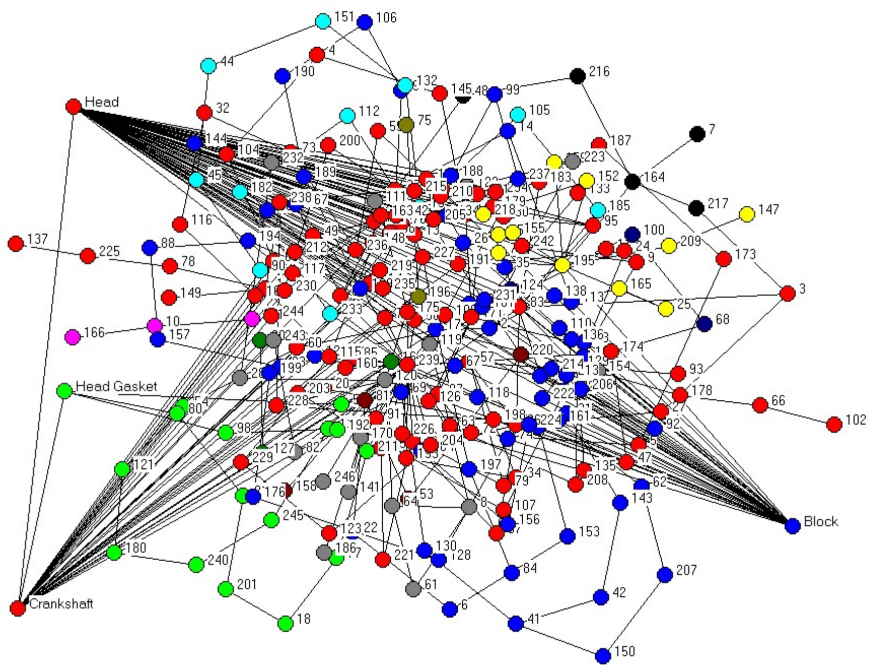
- Picks two pairs of nodes at random and swaps their links so that each node retains its nodal degree
- Usually used to randomize a network
  – Rewire at random, a lot
- Can also be used to change a network's degree correlation or clustering coefficient
  – Rewire but accept only those results that drive r or c in the desired direction
  – Each network has a max and min r that are different from ±1 (papers by Whitney and Alderson, and Li and Alderson)
- Note that this process does not necessarily preserve connectedness, so if this is important, check before accepting each rewiring

Degree-preserving Rewiring Routines

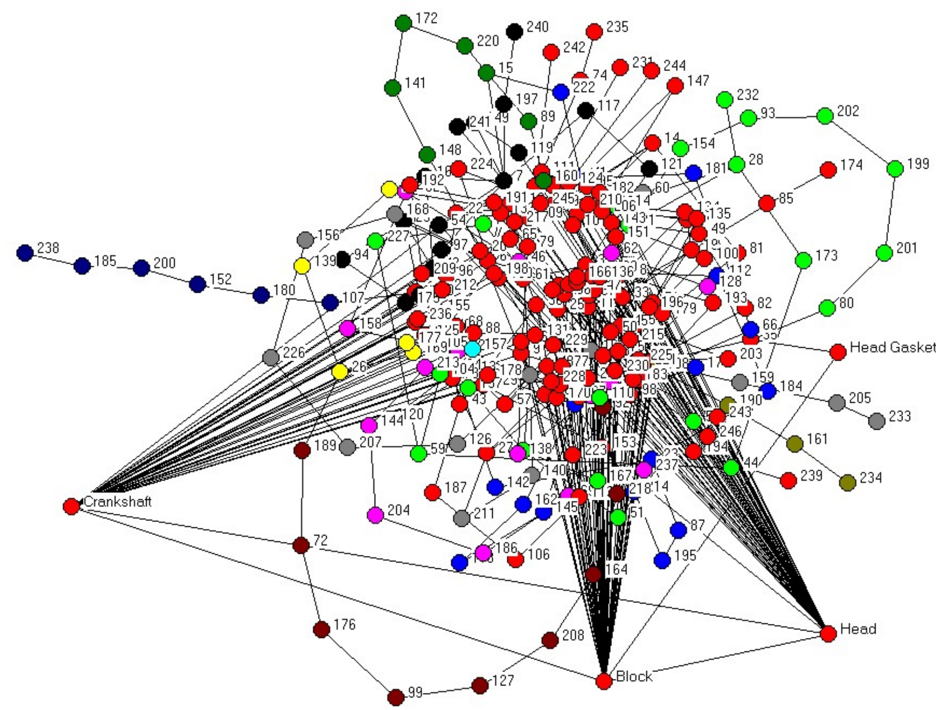
- Maslov-Sneppen routines (the original)
- rgrow, rshrink, cgrow seek to modify the network via directed rewiring to have a different degree correlation or clustering coefficient while preserving the degree sequence and connectedness
  - cgrow is really slow! Use Volz' routine
- rgrowd (does not bother to check for connectedness)
- rgrowdgoal (grows *r* to a desired value called goal, ignores connectedness)
- You can easily write your own to do what you want

Rewired V8 Engine

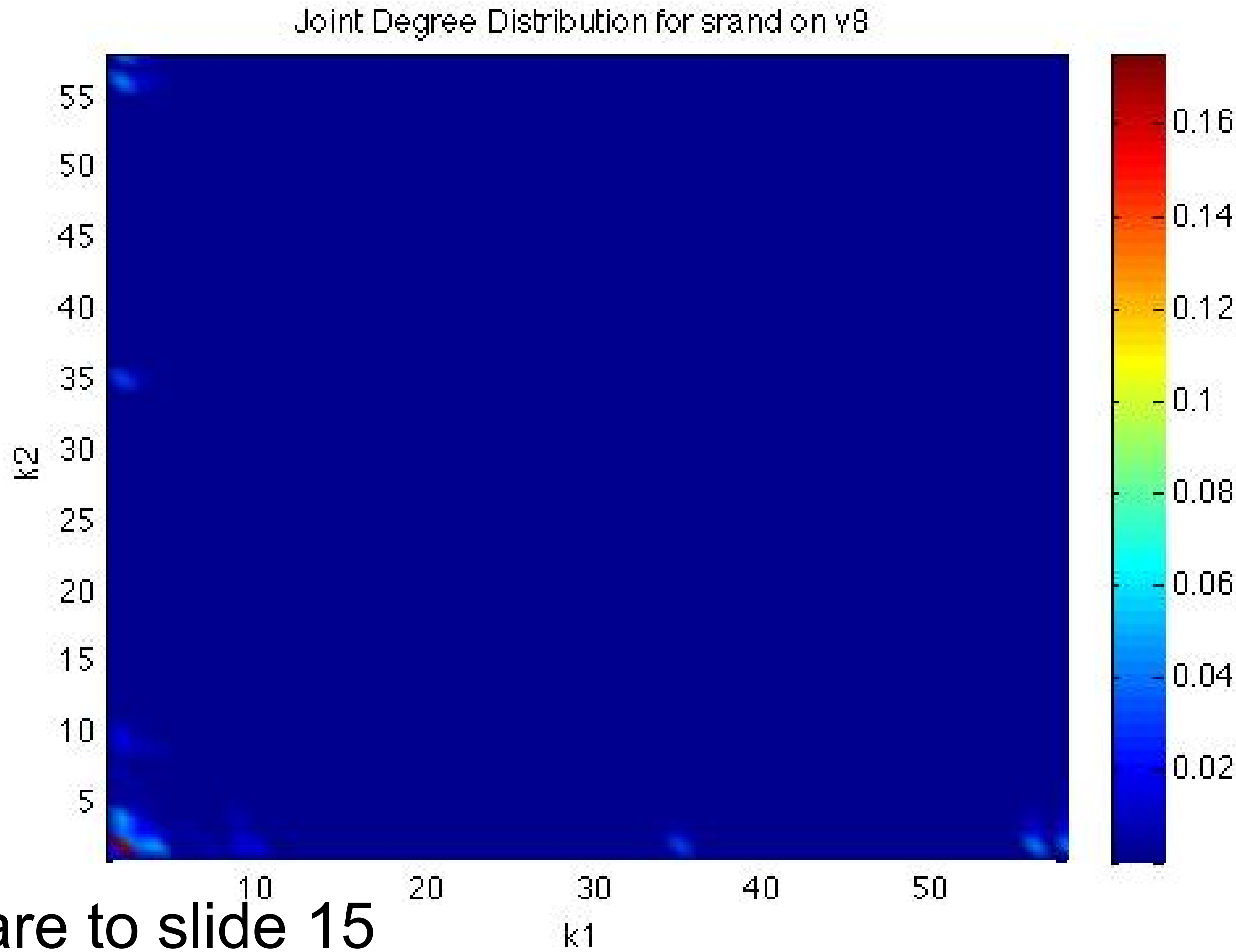
Maslov-Sneppen randomizing



Volz clust reduction



JDD of V8 After Maslov-Sneppen Randomizing

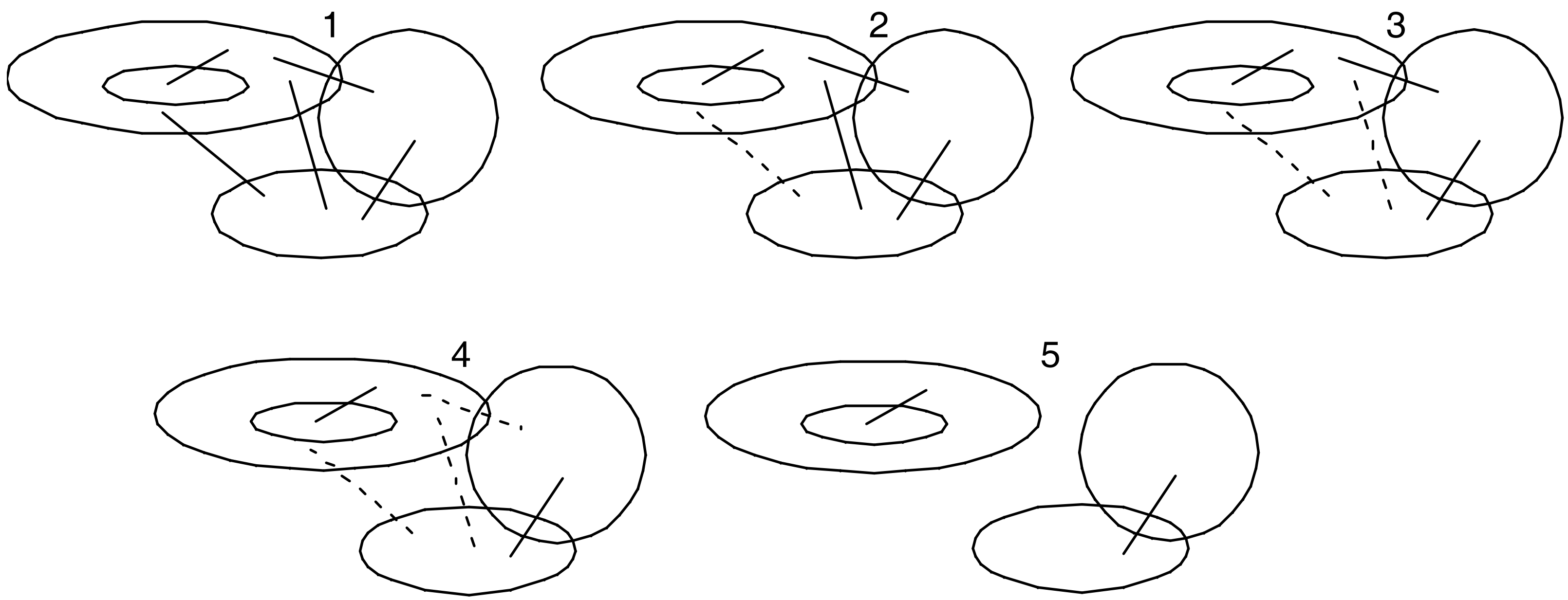


Compare to slide 15

Finding Communities

- Big topic in social network analysis
- Many algorithms exist, based on different principles, several in UCINET
- Recent one based on network flow by Newman and Girvan: M. E. J. Newman and M. Girvan, *Phys. Rev. E* **69**, 026113 (2004).
- Uses the idea of edge betweenness
- Implementation by ESD PhD student Mo-Han Hsieh seems to be more accurate than the implementation in UCINET

Recursive Removal of Highest Betweenness Edge Generates Communities

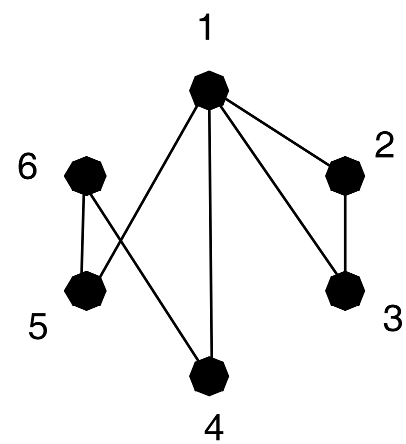


NewmanGirvan.m

```
% This program conducts Newman-Girvan algorithm. Written by Mo-Han Hsieh.
% The input is, A, the adjacency matrix, represented by its edgelist in the file TEST.txt.
% 'Directed' controls whether or not A is directed a network.
% For directed network: Directed=1; for non-directed network: Directed=0
% TarGroupNum is the # of desired communities.
% If TarGroupNum>0, the program will stop at the desired # of communities.
% Output:   QRecord2, dendrogramRecord, and MarkCut
% QRecord2: [mainNum, singletonNum, Q], where mainNum is the # of
% components that have at least two nodes as members, SingletonNum is the #
% of singletons, and Q is the Q defined by Newman-Girvan.
% dendrogramRecord: First row is mainNum, second row is singletonNum, and
% the third row is Q, and the rest rows is the partition of nodes (the same
% format as specified in UCINET).

       A1=load('TEST.txt');
       outputFileName1='Q_resultTEST';
       outputFileName2='dendrogramTEST';
       outputFileName3='CutSequenceTEST';

       m=max(max(A1(:,1:2)));
       % This code builds the adjacency matrix from the edgelist in TEST.txt
       % You can change the code to read A directly and omit reading TEST.txt
       A=zeros(m,m);
       for i=1:size(A1,1)
           A(A1(i,1),A1(i,2))=1;
       end

       Directed=1;
       TarGroupNum=0;
```

Input file TEST.txt

```
>> type TEST.txt

1  2
1  3
1  4
1  5
2  1
2  3
3  1
3  2
4  1
4  6
5  1
5  6
6  4
6  5
```



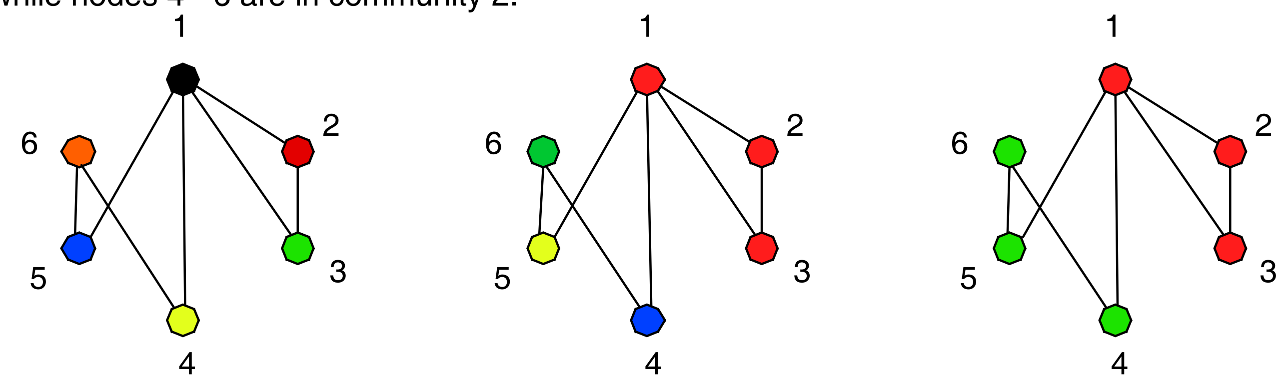
Example Using TEST.txt

Contents of output file dendorgamTEST

0.0000000e+00	1.0000000e+00	2.0000000e+00
6.0000000e+00	3.0000000e+00	0.0000000e+00
-1.8367347e-01	4.0816327e-02	2.0408163e-01
1.0000000e+00	1.0000000e+00	1.0000000e+00
2.0000000e+00	1.0000000e+00	1.0000000e+00
3.0000000e+00	1.0000000e+00	1.0000000e+00
4.0000000e+00	2.0000000e+00	2.0000000e+00
5.0000000e+00	3.0000000e+00	2.0000000e+00
6.0000000e+00	4.0000000e+00	2.0000000e+00

Q is based on density of
Links inside groups compared
To links between groups

There are three candidate partitions of the network, each listed in a column. Reading the first two rows together, one column at a time, we see that the first partition has no main component (zero in row 1) and instead consists of 6 isolated nodes (6 in row 2). The second has one main component and three isolates, while the third has two main components and no isolates. The third row gives Q for each of these, and this is maximum for the third column. The remaining rows contain the community numbers for the 6 respective nodes, in a format suitable for use in UCINET if you want to use Netdraw to draw the network and color the communities. In column 1 we see that each node is in its own community, numbered 1 - 6. In the second column we see that nodes 1 - 3 are in community 1 while 4 - 6 are isolates in communities 2 - 4 respectively. In column 3 we see that nodes 1 - 3 are in community 1 while nodes 4 - 6 are in community 2.



Q = -0.18367 Q = 0.04081 Q = 0.20408

Rich Club Metric

- Measures the extent to which the high degree nodes link to each other
- A subset of Pearson degree correlation since it focuses on the high degree nodes
- Large RCM indicates that high degree nodes link to each other
- Small RCM indicates that they do not
- Base case is a random network with the same degree sequence - ignoring this leads to erroneous conclusions except if the most random equivalent is correlated
- Networks with high RCM can still have $r < 0$
- Ref: paper by Colizza, et al

“Detecting rich-club ordering in complex Networks,” V. COLIZZA, A. FLAMMINI, M. A. SERRANO AND A. VESPIGNANI*
Nature Physics 15 January 2006; doi:10.1038/nphys209

Generating a Graph with a Specified Degree Sequence

- Not any string of numbers qualifies as a degree sequence of a network that is simple and connected
  – Simple: no self-loops, no multiple links between nodes
- Erdos-Gallai theorem tests if a degree sequence is "graphic" (routine isgraphic.m)
- Generating the graph is fraught and often ends up incomplete or disconnected, or else it has some self-loops and multiple edges between nodes

Random Graph Realization Summary

Function⇒ **Routine or folder ⇓**	**Generate the degree sequence**	**Generate the graph from the degree sequence**	**Remarks**
degree_dist	Use it to generate most distributions except power law	No	First few lines of random_graph
random_graph	Most distributions except power law	Yes	Graph generation is slow for $n >$ 100 - 200
erdosRenyi in folder randGraphs	Watts-strogatz grids	Yes plus a plot	Only one type of graph
sfng in folder Barabasi-Albert	Power law with $2 < k < 3$ typically	As above	As above
Folder Volz	No	Generates a symmetric edge list	Can choose the clustering coeff
buildSmax	No	Builds graph with max positive degree correlation	Only one type of graph

Random (Poisson) Networks

- randmatrix(n, p);
- Since p=z/n, you can write randmatrix(n, z/n);
- This generates the adjacency matrix for a random network of n nodes having probability p of a link between any pair of nodes chosen at random
- The degree distribution is poisson with average=z, clustering coefficient ~ p and r ~ 0
- Original theory due to Erdös and Renyi so these are often called ER random graphs

random_graph.m

```
% Random graph construction routine with various models
% Gergana Bounova, October 31, 2005

function [adj] = random_graph(N,p,E,distribution,fun,degrees)

% INPUTS:
% N - number of nodes
% p - probability, 0<=p<=1
% E - fixed number of edges
% distribution - probability distribution: use the
%               "connecting-stubs model"
%               generation model
% choices are uniform, normal, binomial, exponential,geometric
% set parameters by modifying the code
% fun - customized pdf function, used only if distribution =
%      'custom'
% degrees - particular degree sequence, used only if distribution =
%         'sequence'

% OUTPUTS: adj - adjacency matrix of generated graph (symmetric)
% Only the first argument is needed, but if any number of arguments is
%  provided, all up to that number must be provided, even though
%  only N and the kind of distribution would be used.  Others, like E,
%  will be ignored
```

degree_dist.m

```
function [Nseq] = degree_dist(N,p,distribution)
% Random graph degree sequence construction routine with various models
% Gergana Bounova, October 31, 2005, modified by Whitney 1-8-08
% INPUTS:
% N - number of nodes
% p - probability, 0<=p<=1
% distribution - probability distribution name, used below
% choices are 'uniform', 'binomial', 'normal', 'exponential'
% change parameters in the code below to get mean, variance, etc

% OUTPUTS: NSeq - degree sequence drawn from the specified distribution
```

Example Calls to random_graph

```
random_graph(10)
random_graph(10,0.1,20)
random_graph(10,0,0,'normal')
random_graph(10,0,0,'custom',@mypdf)
degs = [3 1 1 1];
random_graph(10,0,0,'custom',@mypdf,degs)
```


Volz' Algorithm

- Originally intended to generate a graph with specified degree sequence and specified clustering
- Getting the right clustering is difficult
- Volz' method is fast and can be used to generate a graph with any degree sequence and zero clustering
- It is in Java and must be executed from the operating system
- But the Matlab command window is an operating system shell if you use "!" to start the command

Script for Volz Routine

```
% network_generator_script
% script to generate random networks with given degree sequence
% Java executable RandomClusteringNetwork.jar must be in your matlab
% directory
N=100
p=0.1
E=10
distribution='normal'
fun=1
degrees=1
stop=1

Nseq = degree_dist(N,p,E,distribution,fun,degrees,stop);
Nseqabs=abs(Nseq); %protect against negative values
Nseqint=int16(Nseqabs); %Volz routine requires integers

dlmwrite('degdist.txt',Nseqint,'\t') %Volz routine requires tab delimited input
!java -jar  RandomClusteringNetwork.jar degdist.txt 100 .001 output.txt % n = 100, desired clust =
% if you use 0.0 for desired clust the program will crash!
outputedges=dlmread('output.txt'); %Volz routine generates a symmetric edge list
outputadj=adjbuilde(outputedges);
kvoutputadj=kvec(outputadj);
khatoutputadj=khat(outputadj)
sigmaoutputadj=stdev(kvoutputadj)
```

PHYSICAL REVIEW E **70**, 056115 (2004)

Random networks with tunable degree distribution and clustering

Erik Volz
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(Received 4 June 2004; published 17 November 2004)

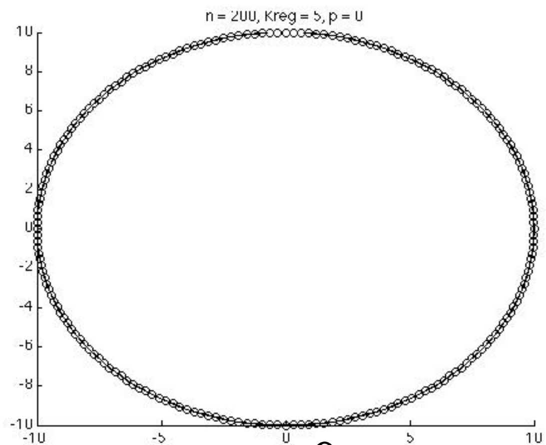
erdosRenyi.m

- Actually this routine makes a Watts-Strogatz random graph, not a Poisson (ER) random graph
- It starts from a ring mesh where $k = Kreg$ at each node (only even values of k should be used)
- With probability p it unhooks one end of a link and puts it down on another node
- This is not the same p as in randmatrix
- This kind of rewiring preserves the networks' z but does not preserve the degree sequence

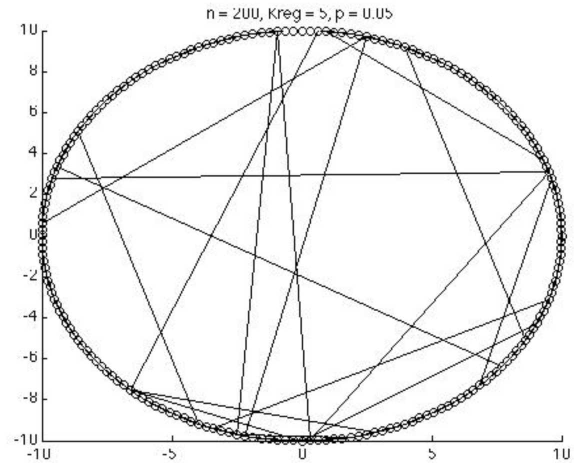
Watts-Strogatz Small World Generator

```
function [G]=erdosRenyi(nv,p,Kreg)
%Function [G]=edosRenyi(nv,p,Kreg) generates a random graph based on
%the Erdos and Renyi algoritm where all possible pairs of 'nv' nodes are
%connected with probability 'p'. It does this by creating a connected
%regular grid with k = Kreg at every node and then rewires.  It does not
%protect against disconnecting the network or isolating nodes.
%
% Inputs:
%   nv - number of nodes
%   p  - rewiring probability
%   Kreg - initial node degree of for regular graph (use 1 or even numbers)
%
% Output:
%   G is a structure implemented as data structure in this as well as other
%   graph theory algorithms.
%   G.Adj   - is the adjacency matrix (1 for connected nodes, 0 otherwise).
%   G.x and G.y -   are row vectors of size nv wiht the (x,y) coordinates of
%                each node of G.
%   G.nv    - number of vertices in G
%   G.ne    - number of edges in G
%  Created by Pablo Blinder.
```

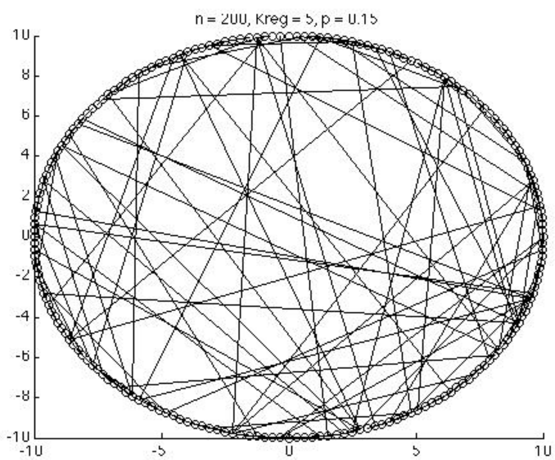
Watts-Strogatz Examples Using erdosRenyi Code, *n* = 200, *Kreg* = 4



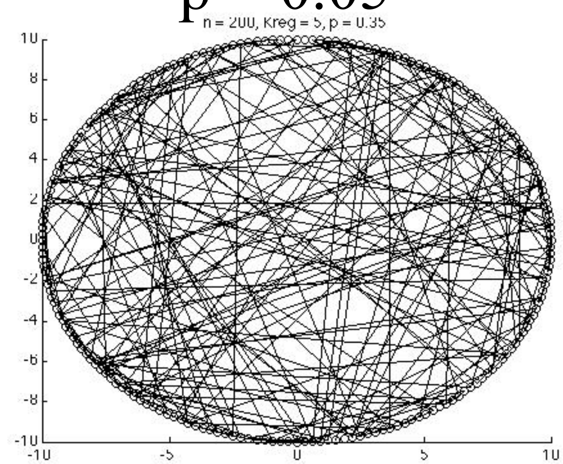
p = 0



p = 0.05



p = 0.15



p = 0.35

Watts-Strogatz Model

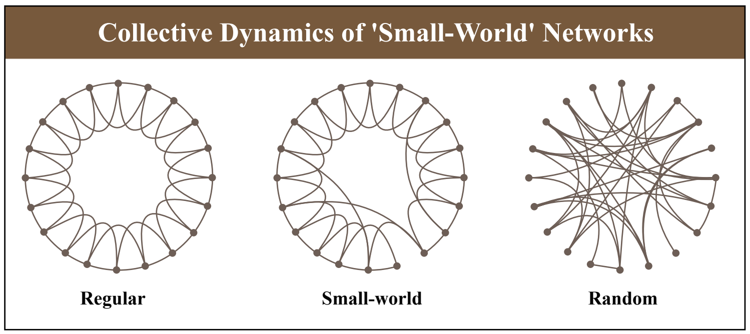
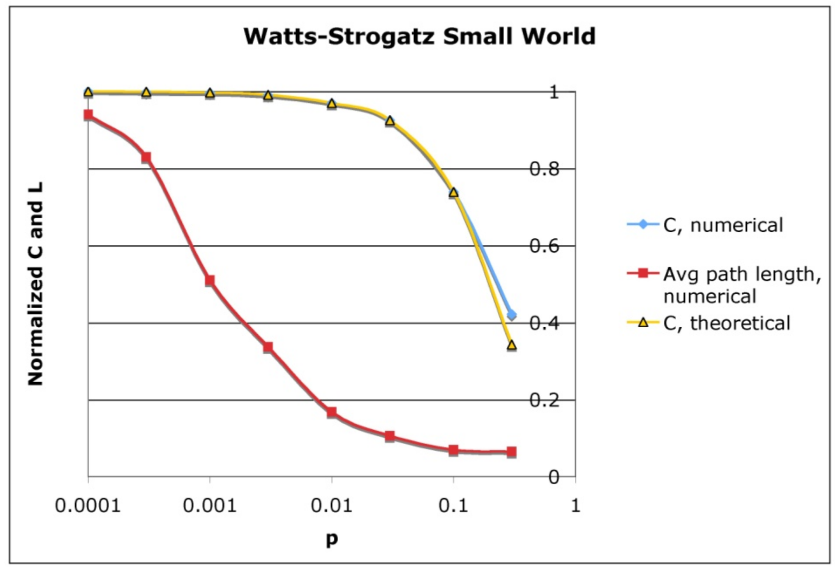


Image by MIT OpenCourseWare.



$$< \ell > = \frac{n}{2z} = \frac{n}{2 < k >}$$

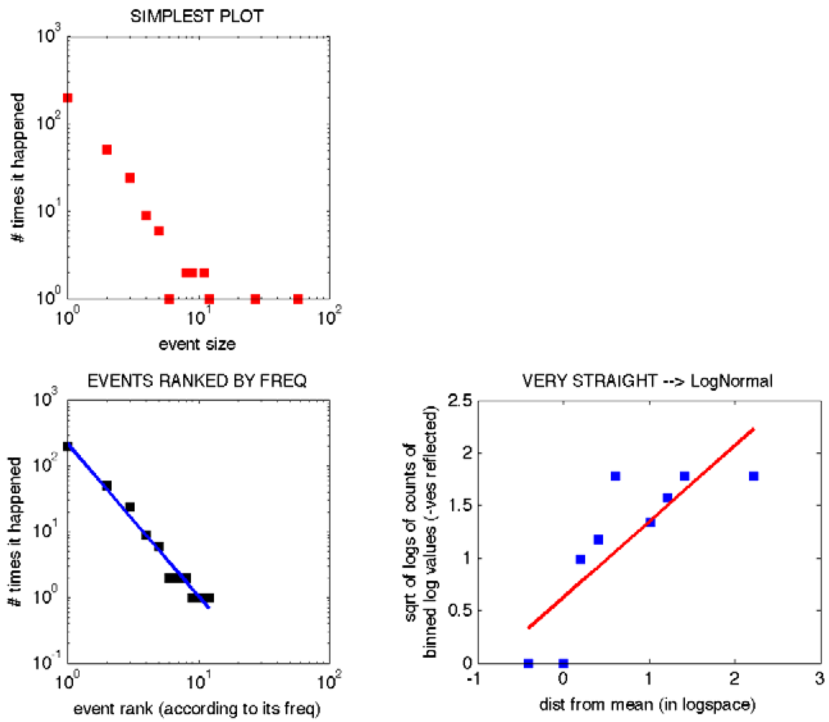
when $p \sim 0$

$$C = \frac{3(z/2-1)}{2(z-1)}(1-p)^3$$

SFNG

- Text from the “read me:”
- B-A Scale-Free Network Generation and Visualization
- By Mathew Neil George
- The *SFNG* m-file is used to simulate the B-A algorithm and returns scale-free networks of given sizes.
- Here is a small example to demonstrate how to use the code. This code creates a seed network of 5 nodes, generates a scale-free network of 300 nodes from the seed network, and then performs the two graphing procedures.
- seed =[0 1 0 0 1;1 0 0 1 0;0 0 0 1 0;0 1 1 0 0;1 0 0 0 0]
- Net = SFNG(300, 1, seed);
- CNet(Net) % draws the graph
- diagnose_matrix(Net,20) % Gergana's routine. Tells you the exponent
- %PL_Equation = PLplot(Net) neets "fit"

SFNG Output



Fit for power law: -2.3227x + 5.4084

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