Inventory Management
Probabilistic Demand

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Assumptions: Probabilistic Demand

- Demand
  - Constant vs **Variable**
  - Known vs **Random**
  - **Continuous** vs Discrete
- Lead time
  - Instantaneous
  - **Constant** or Variable (deterministic/stochastic)
- Dependence of items
  - **Independent**
  - Correlated
  - Indentured
- Review Time
  - **Continuous**
  - Periodic
- Number of Echelons
  - **One**
  - Multi (>1)
- Capacity / Resources
  - **Unlimited**
  - Limited / Constrained
- Discounts
  - **None**
  - All Units or Incremental
- Excess Demand
  - None
  - **All orders are backordered**
  - **All orders are lost**
  - Substitution
- Perishability
  - **None**
  - Uniform with time
- Planning Horizon
  - Single Period
  - Finite Period
  - **Infinite**
- Number of Items
  - **One**
  - Many
- Form of Product
  - **Single Stage**
  - Multi-Stage

Key Questions

- What are the questions I should ask to determine the type of inventory control system to use?
  - How important is the item?
  - Should review be periodic or continuous?
  - What form of inventory policy should I use?
  - What cost or service objectives should I set?

How important is the item?

- Standard ABC analysis
  - A Items
    - Very few high impact items are included
    - Require the most managerial attention and review
    - Expect many exceptions to be made
  - B Items
    - Many moderate impact items (sometimes most)
    - Automated control w/ management by exception
    - Rules can be used for A (but usually too many exceptions)
  - C Items
    - Many if not most of the items that make up minor impact
    - Control systems should be as simple as possible
    - Reduce wasted management time and attention
    - Group into common regions, suppliers, end users
- But – these are arbitrary classifications

Continuous or Periodic Review?

- Periodic Review
  - Know stock level only at certain times
  - Review periods are usually scheduled and consistent
  - Ordering occurs at review
- Pros / Cons
  - Coordination of replenishments
  - Able to predict workload
  - Forces a periodic review

- Continuous Review
  - Is continuous really continuous?
  - Transactions reporting
  - Collecting information vs. Making decision
- Pros / Cons
  - Replenishments made dynamically
  - Cost of equipment
  - Able to provide same level of service with less safety stock

- Notation

s = Order Point	S = Order-up-to Level	L = Order Lead Time
Q = Order Quantity	R = Review Period	IOH= Inventory on Hand

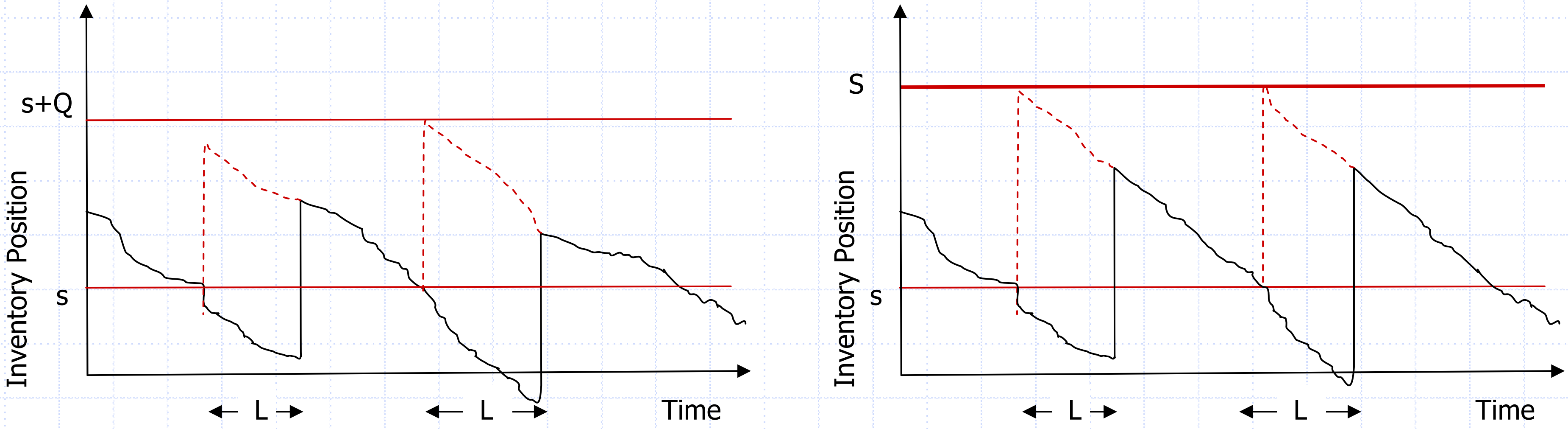
IP = Inventory Position

= (IOH) + (Inv On Order) – (Backorders) – (Committed)

What form of inventory policy?

Continuous Review (R=0)

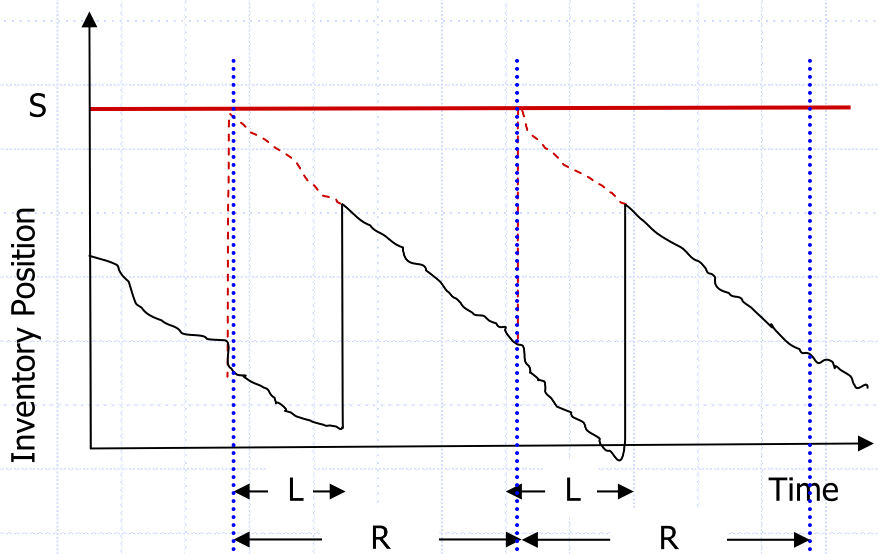
- Order-Point, Order-Quantity (s, Q)
  - Policy: Order Q if IP ≤ s
  - Two-bin system
- Order-Point, Order-Up-To-Level (s, S)
  - Policy: Order (S-IP) if IP ≤ s
  - Min-Max system



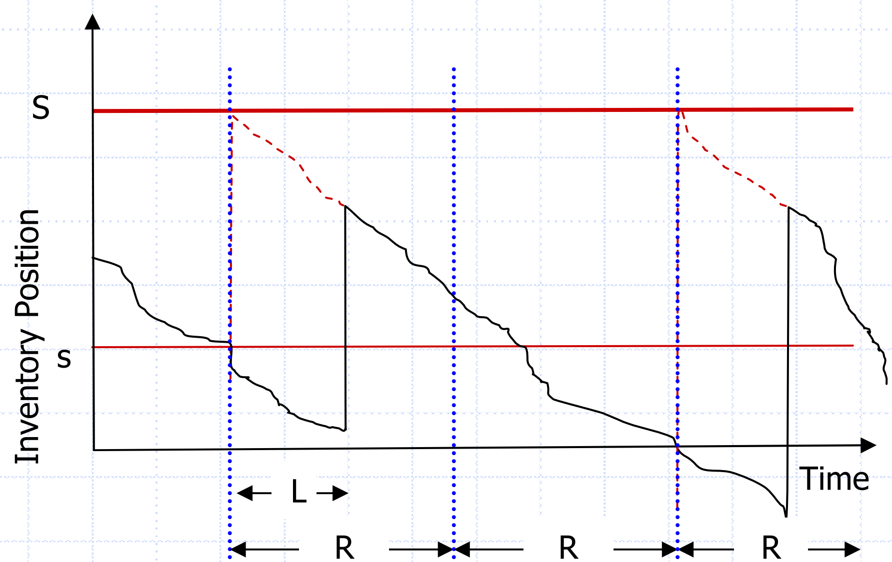
What form of inventory policy?

Periodic Review (R>0)

- Order-Up-To-Level (R, S)
  - Policy: Order S-IP every R time periods
  - Replenishment cycle system



- Hybrid (R, s, S) System
  - Policy: Order S-IP if IP ≤ s every R time periods, if IP>s then do not order
  - General case for many policies



What form of inventory policy?

- No hard and fast rules, but some rules of thumb

Type of Item,	Continuous Review	Periodic Review
A Items	(s, S)	(R, s, S)
B Items	(s,Q)	(R, S)
C Items		Manual ~ (R,S)

Determining s in (s,Q) System

- Coverage over lead time
  - Expected demand over lead time
  - Safety (buffer stock)
- Procedure:
  - Find Safety Stock (SS) by specifying a k
  - Find s by adding SS to expected demand over leadtime

$$s = x_L + k\sigma_L$$

Reorder Point

Forecast demand over the lead time

Safety Stock

k = SS factor

σ_L = RMSE

- Parameters depend on cost & service objectives

What cost and service objectives?

1. Common Safety Factors Approach
   - Simple, widely used method
   - Apply a common metric to aggregated items
2. Cost Minimization Approach
   - Requires costing of shortages
   - Find trade-off between relevant costs
3. Customer Service Approach
   - Establish constraint on customer service
   - Definitions in practice are fuzzy
   - Minimize costs with respect to customer service constraints
4. Aggregate Considerations
   - Weight specific characteristic of each item
   - Select characteristic most "essential" to firm

Framework for (s, Q) Systems

- Cycle Stock
  - Determine best Q
  - Usually from EOQ
- Safety Stock
  - Pick type of cost or service standard
    - If service, then use decision rule for setting k
    - If cost, then minimize total relevant costs to find k
  - Calculate safety stock as $k\sigma_L$
- Total Cost:

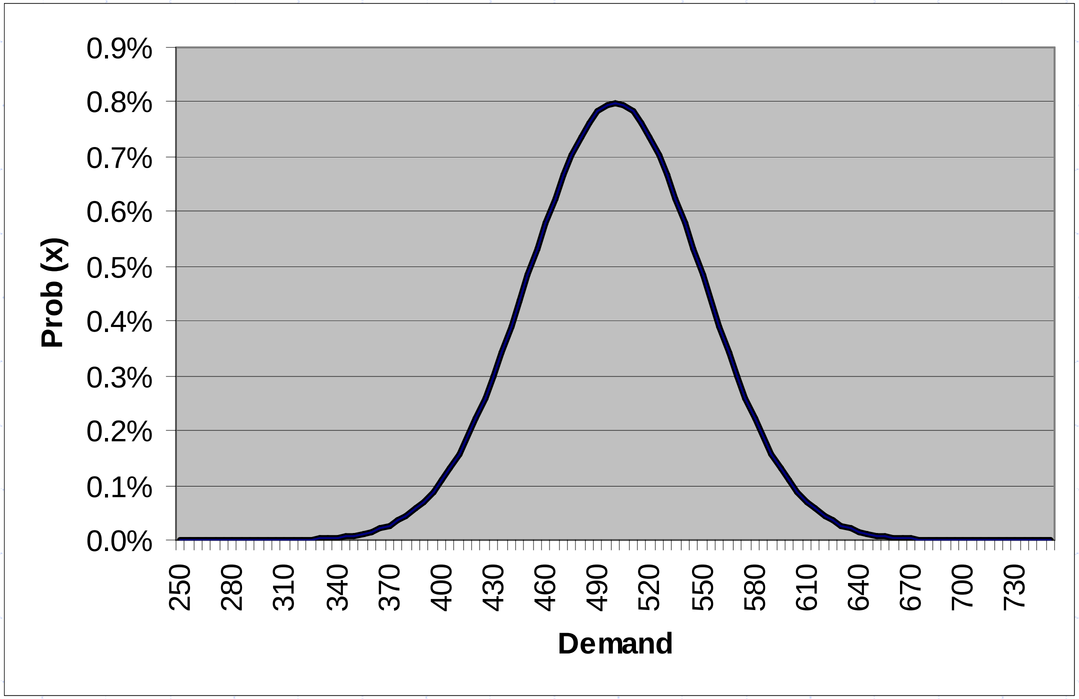
$$TC = vD + A\left(\frac{D}{Q}\right) + vr\left(\frac{Q}{2} + k\sigma_L\right) + C_{StockOutType} P[StockOutType]$$

Framework for (s,Q) Systems

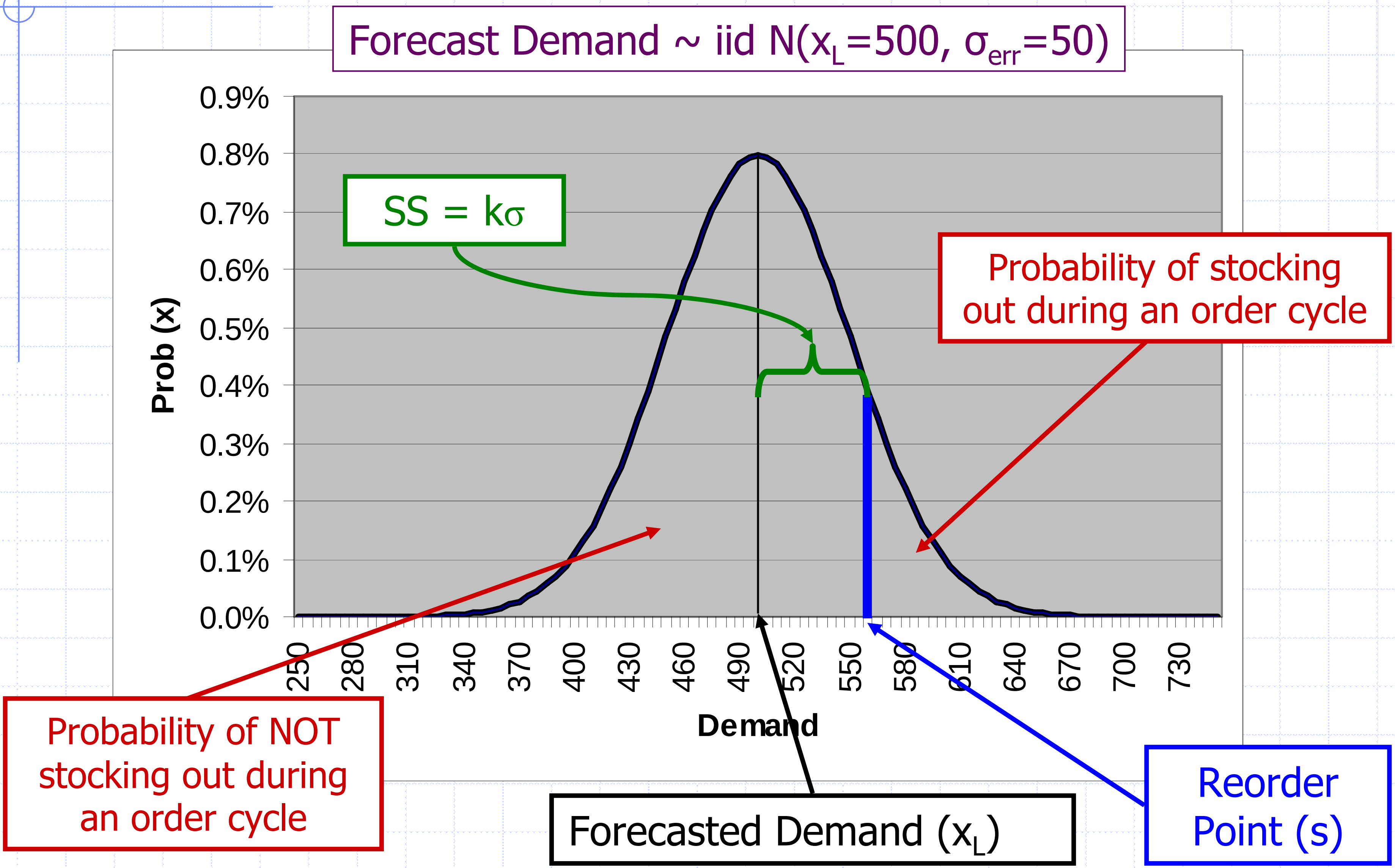
Stockout Types	Key Element	Cost	Service
Event based	Probability of a stock out event	B_1(Prob[SO])(D/Q)	P_1=1-Prob[SO]
# of Units Short	Expected # units short	$(B_2 v)(\sigma_L G_u(k))(D/Q)$	P_2=ItemFillRate =1- $(\sigma_L G_u(k)/Q)$
Units Short per Time	Expected duration time for each unit stocked out	$(B_3 v)(\sigma_L G_u(k) d_{SO})(D/Q)$ Where d_{SO}=avg duration of stockout	
Line Items Short	Expected number of lines shorted	$(B_4 v)(\sigma_L G_u(k)/z)(D/Q)$ where z=avg items / order	

Cycle Service Level (CSL or P_1)

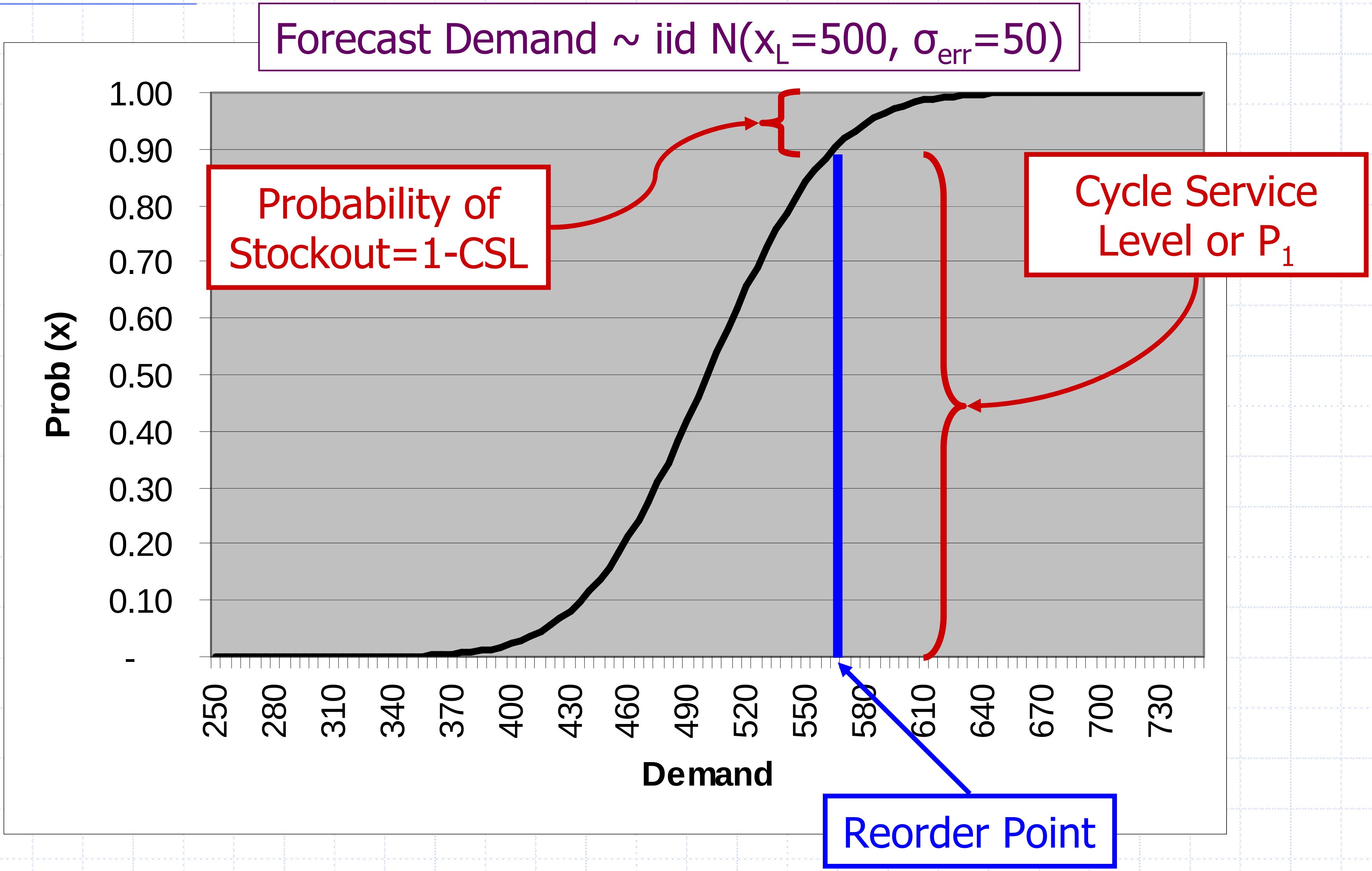
- Cycle Service Level
  - Probability of no stockouts per replenishment cycle
  - Equal to one minus the probability of stocking out
  - $= 1 - P[\text{Stockout}] = 1 - P[x_L > s] = P[x_L \leq s]$



Finding P[Stockout]



Cumulative Normal Distribution



Finding CSL from a given k

If I select a k=0.42

Using a Table of Cumulative Normal Probabilities . . .

K	0.00	0.01	0.02	0.03	0.04
0.0	0.5000	0.5040	0.5080	0.5120	0.5160
0.1	0.5398	0.5438	0.5478	0.5517	0.5557
0.2	0.5793	0.5832	0.5871	0.5910	0.5948
0.3	0.6179	0.6217	0.6255	0.6293	0.6331
0.4	0.6554	0.6591	0.6628	0.6664	0.6700
0.5	0.6915	0.6950	0.6985	0.7019	0.7054
[illegible]	[illegible]	[illegible]	[illegible]	.7357	0.7389
[illegible]	[illegible]	[illegible]	[illegible]	.7673	0.7704
[illegible]	[illegible]	[illegible]	[illegible]	.7967	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	.8238	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	.8485	[illegible]

. . . then my Cycle Service Level is this value.

- From SPP (Table B.1 pp 724-734)
  - Select k factor (first column)
  - Prob[Stockout] = value in the $p_{u\geq}(k)$ column
  - CSL = 1- $p_{u\geq}(k)$
- In Excel, use the function
  - CSL=NORMDIST(s, x_L, σ_L, 1) where s= $x_L + k\sigma_L$
  - CSL=NORMSDIST(k)

k Factor versus Cycle Service Level

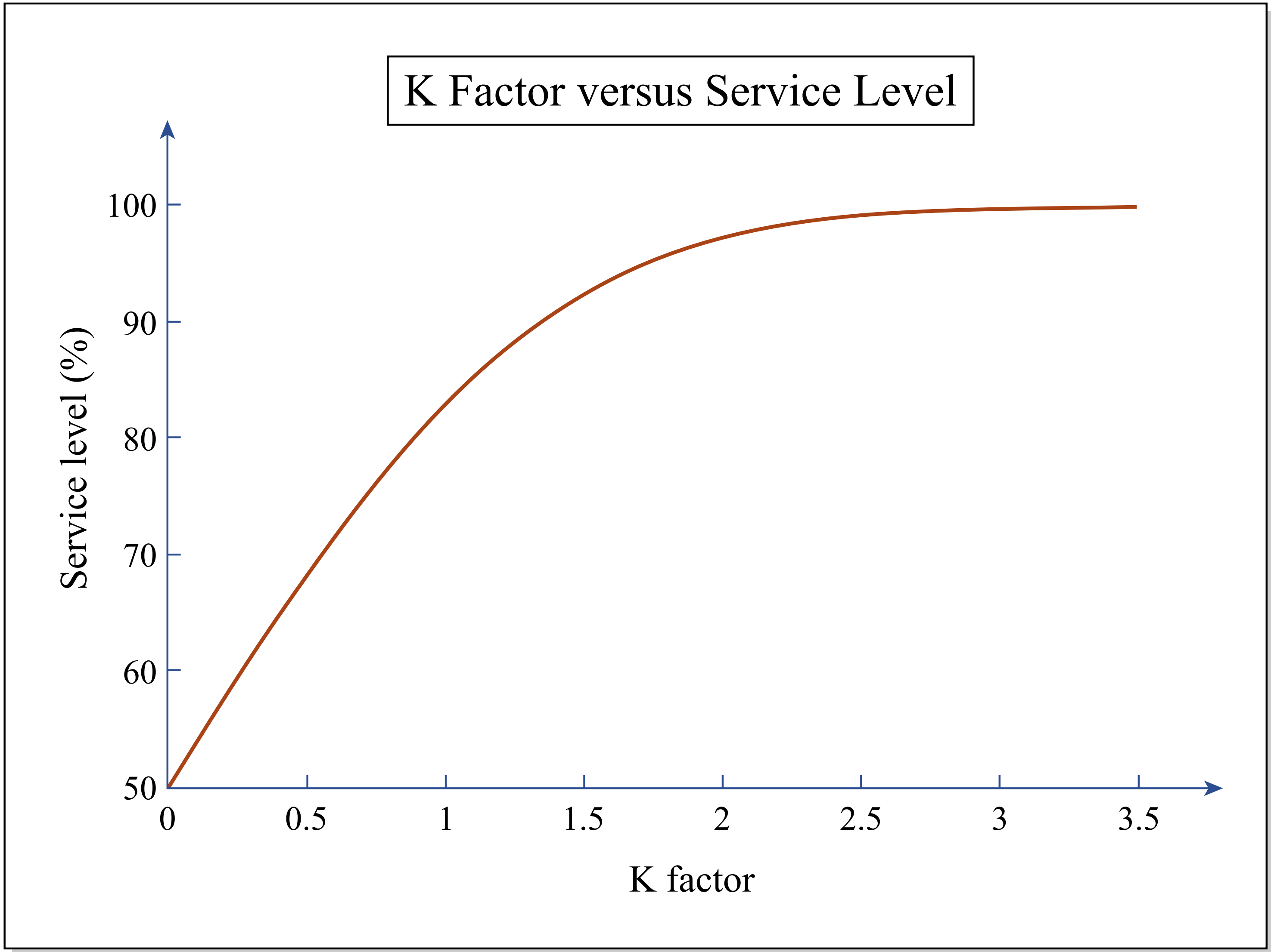


Figure by MIT OCW.

Example: Setting SS and s

- Given
  - Average demand over time is considered constant
  - Forecast of demand is 13,000 units a year ~ iid Normal
  - Lead time is 2 weeks
  - RMSE of the forecast = 1,316 units per year
  - EOQ = 228 units (A=50 $/order, r=10%, v=250 $/item)
- Find
  - Safety stock and reorder point, s, for the following cycle service levels:
    - CSL=.80
    - CSL=.90
    - CSL=.95
    - CSL=.99

Quick Aside on Converting Times

- How do I convert expected values and variances of demand from one time period to another?
- Suppose we have two periods to consider:
  - S = Demand over short time period (e.g., week)
  - L = Demand over long time period (e.g., year)
  - n = Number of short periods within a long (e.g., 52)

- Converting from Long to Short
  - E[S] = E[L]/n
  - VAR[S] = VAR[L]/n so that $\sigma_S = \sigma_L/\sqrt{n}$
- Converting from Short to Long
  - E[L] = nE[S]
  - VAR[L] = nVAR[S] so that $\sigma_L = \sqrt{n}\,\sigma_S$

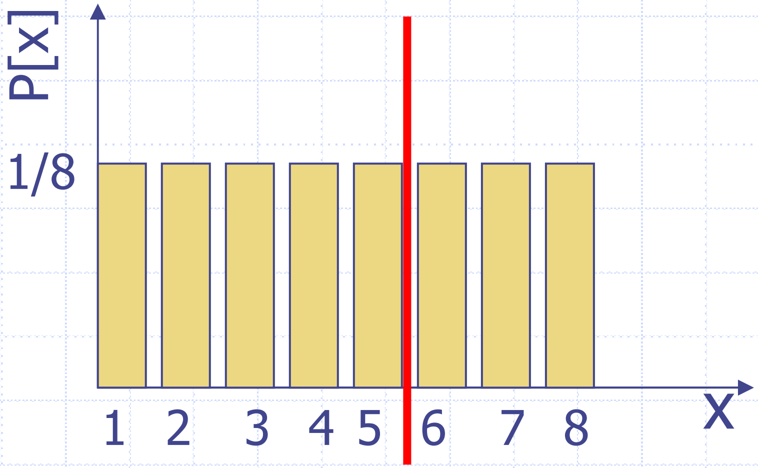
Item Fill Rate (P_2) Metric

- Item Fill Rate (P_2)
  - Fraction of demand filled from IOH
  - Need to find the expected number of items that I will be short for each cycle
    - Expected Units Short E[US]
    - Expected Shortage per Replenishment Cycle (ESPRC)
  - More difficult than CSL – need to find a partial expectation for units short

$$FillRate = \frac{OrderQuantity - E[UnitsShort]}{OrderQuantity}$$

Finding Expected Units Short

- Find the expected number of units short, E[US], during a replenishment cycle
- Use Loss Function – widely used in inventory theory
- L(k) = expected amount that random variable X exceeds a given threshold value, k.



What is L[k] if k=5?

Interpretation:
If my demand is ~U(1,8) and I have a safety stock of 5 then I can expect to be short 0.75 units each service cycle

Finding Expected Units Short

Consider both continuous and discrete cases

Looking for expected units short per replenishment cycle.

$$E[US] = \sum_{x=k}^{\infty} (x-k)\, p[x]$$

$$E[US] = \int_{k}^{\infty} (x_o - k)\, f_x(x_o) dx_o$$

For normal distribution, $E[US] = \sigma_L G(k)$

Where G(k) = Unit Normal Loss Function

In SPP,

$G(k) = G_u(k) = f_x(x_O) - k*Prob[x_O \geq k])$

Derived in SPP p. 721, in tables B.1

In Excel,

NORMDIST(k,0,1,0) – k(1 - NORMDIST(k,0,1,1))

Item Fill Rate (IFR or P_2)

- Procedure: Relate k to desired IFR
  - Find k that satisfies rule
    - Solve for G[k]
    - Use table or Excel to find k
  - Find reorder point s
    - $s = x_L + k\sigma_L$

$$IFR = \frac{Q - E[US]}{Q} = 1 - \frac{E[US]}{Q}$$

$$IFR = 1 - \frac{\sigma_L G[k]}{Q}$$

$$G[k] = \frac{Q}{\sigma_L}(1 - IFR)$$

- Example
  - Average demand over time is considered constant
  - Forecast of demand is 13,000 units a year ~ iid Normal
  - Lead time is 2 weeks
  - RMSE of the forecast = 1,316 units per year
  - EOQ = 228 units (A=50 $/order, r=10%, v=250 $/item)
- Find
  - Safety stock and reorder point, s, for the following item fill rates:
    - IFR=.80, .90,.95, and 0.99

Compare CSL versus IFP

- IFR usually much higher than CSL for same SS
- IFR depends on both s and Q while CSL is independent of all product characteristics
- Q determines the number of exposures for an item

Pct	SS CSL	SS IFR
99%	601	513
95%	423	348
90%	330	252
80%	217	148

Cost per Stockout Event (B_1)

- Consider total relevant costs
  - Order Costs – no change from EOQ
  - Holding Costs – add in Safety Stock
  - StockOut Costs product of:
    - Cost per stockout event (B_1)
    - Number of replenishment cycles
    - Probability of a stockout per cycle

$$TRC = OrderCosts + HoldingCosts + StockOutCosts$$

$$TRC = A\left(\frac{D}{Q}\right) + \left(\frac{Q}{2} + k\sigma_L\right)vr + B_1\left(\frac{D}{Q}\right)p_{u\geq}(k)$$

- Solve for k that minimizes total relevant costs
  - Use solver in Excel
  - Use decision rules

Cost per Stockout Event (B_1)

- Decision Rule
  - If Eqn 7.19 is true
    - Set k to lowest allowable value (by mgmt)
  - Otherwise set k using Eqn 7.20

$$(Eqn 7.19) \quad \frac{DB_1}{\sqrt{2\pi} Q v \sigma_L r} < 1$$

$$(Eqn 7.20) \quad k = \sqrt{2 \ln\left(\frac{DB_1}{\sqrt{2\pi} Q v \sigma_L r}\right)}$$

Cost per Unit Short (B_2)

- Consider total relevant costs
  - Order Costs – no change from EOQ
  - Holding Costs – add in Safety Stock
  - StockOut Costs product of:
    - Cost per item stocked out (B_2)
    - Estimated number units short
    - Number of replenishment cycles

$$TRC = OrderCosts + HoldingCosts + StockOutCosts$$

$$TRC = A\left(\frac{D}{Q}\right) + \left(\frac{Q}{2} + k\sigma_L\right)vr + B_2 v\sigma_L G_u(k)\left(\frac{D}{Q}\right)$$

- Solve for k that minimizes total relevant costs
  - Use solver in Excel
  - Use decision rules

Cost per Unit Short (B_2)

- Decision Rule
  - If Eqn 7.22 is true
    - Set k to lowest allowable value (by mgmt)
  - Otherwise set k using Eqn 7.23

$$(Eqn7.22) \quad \frac{Qr}{DB_2} > 1$$

$$(Eqn7.23) \quad p_{u\geq}(k) = \frac{Qr}{DB_2}$$

Example

- You are setting up inventory policy for a Class B item. The annual demand is forecasted to be 26,000 units with an annual historical RMSE +/- 2,800 units. The replenishment lead time is currently 4 weeks. You have been asked to establish an (s,Q) inventory policy.
- Other details: It costs $12,500 to place an order, total landed cost is $750 per item, holding cost is 10%. Items come in cases of 100 each.
- What is my policy, safety stock, and avg IOH if . . .
  1. I want to have a CSL of 95%?
  2. I want to achieve an IFR of 95%?
  3. I estimate that the cost of a stockout per cycle is $50,000?
  4. I estimate that the cost of a stockout per item is $75?

Questions?
Comments?
Suggestions?