MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.097 (UG) Fundamentals of Photonics
6.974 (G) Quantum Electronics

Spring 2006

Final Exam

Time: May 23*, 2006, 1:30-4:30pm*

Problems marked with (Grad) are for graduate students only.

- This is a closed book exam, but three 81/2"x11" sheets (both sides) are allowed.
- At the end of the booklet there is a collection of equations you might find helpful for the exam.
- Everything on the notes must be in your original handwriting (i.e. material cannot be Xeroxed).
- You have 3 hours for this exam.
- There are 8 problems for undergraduate and 9 problems for graduate students on the exam with the number of points for each part and the total points for each problem as indicated. Note, that the problems do not all have the same total number of points.
- Some of the problems have parts for graduate students only. Undergraduate students solving these problems can make these additional points and compensate eventually for points lost on other problems.
- Make sure that you have seen all 29 numbered sides of this answer booklet.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- We tried to provide ample space for you to write in. However, the space provided is not an indication of the length of the explanation required. Short, to the point, explanations are preferred to long ones that show no understanding.
- Please be neat-we cannot grade what we cannot decipher.

All work and answers must be in the space provided on the exam booklet. You are welcome to use scratch pages that we provide but when you hand in the exam we will not accept any pages other than the exam booklet.

Exam Grading

In grading of the exams we will be focusing on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess, from your work, your level of understanding. On each part of an exam question we will also indicate the percentage of the total exam grade represented by that part, and your numerical score on the exam will then be calculated accordingly.

Our assessment of your level of understanding will be based upon what is given in your solution. A correct answer with no explanation will not receive full credit, and may not receive much-if any. An incorrect final answer having a solution and explanation that shows excellent understanding quite likely will receive full (or close to full) credit.

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No work on this page will be evaluated.

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Final Exam

Time: May 23*, 2006, 1:30-4:30pm*

Full Name: ____

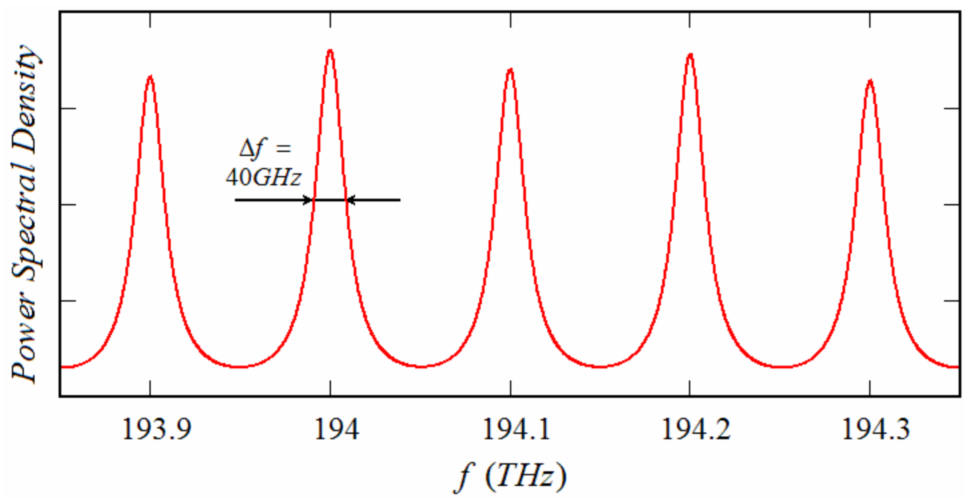
Are you taking 6.974 ____ or 6.097 ____ ?

		Your Points	Max points (undergrad.)	Max points (grad.)
1	Fabry-Perot Filter		20	20
2	ABCD Matrices		12	12
3	Gaussian Beams/Resonators		5	5
4	Polarization of Light		18	18
5	Harmonic Oscillator		20	20
6	Operators/Commutators		10	10
7	Two-Level Atom		15	15
8	Laser		25	25
9	Three-Level System		-	25
Total			**125**	**150**

Problem 1: Fabry-Perot Filter

(20 points)

In most fiber optics transmission systems a single fiber is used for transmission of multiple channels, each of which has a different carrier frequency. Suppose that you have a system with 40 channels; each channel has 40 GHz bandwidth and these channels are spaced by 100 GHz. A part of this spectrum is illustrated below.



At the output of the fiber, each of these channels must be separated from the others before information carried by this channel can be used. Suppose we want to use a Fabry-Perot resonator as a filter to select the single channel at 194 THz. The two mirrors of the Fabry-Perot are identical and are separated by material with a refractive index of $n = 1.5$.

(a) *(5 points)* What is the requirement for the free spectral range of the Fabry-Perot filter to be used for selecting 1 of these 40 channels?

(b) *(15 point)* Choose the length L of the Fabry-Perot cavity and the reflectivity of the mirrors, R, that can be used for extracting the channel at **194 THz**. We have the constraint that R cannot exceed 99%. If you think that the answer to this problem is not unique, give a combination of L and R that will work.

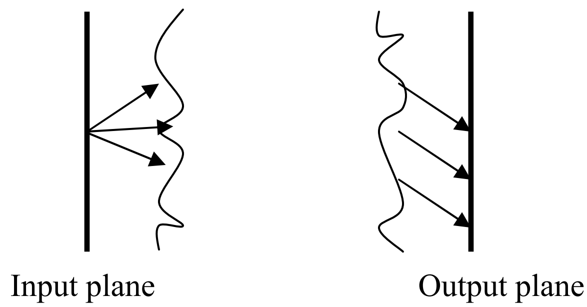
(b) continued

Problem 2: ABCD Matrices

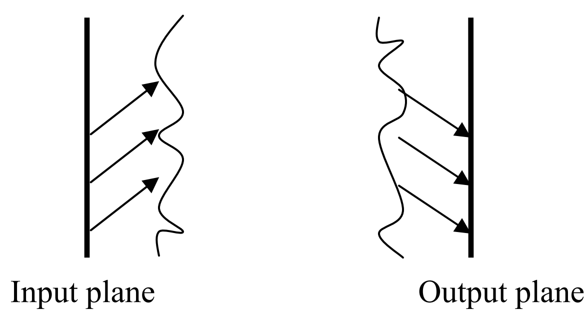
(12 points)

In each figure below, we have an input plane and an output plane separated by some unknown optical elements. The arrows in these figures denote optical rays. What can you say about the matrix elements of each optical system's ABCD matrix?

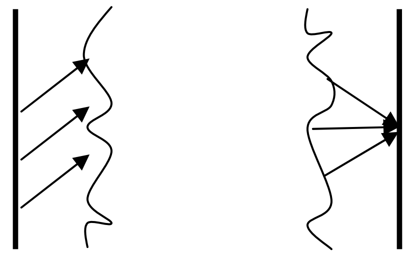
(a) (3 points)



(b) (3 points)

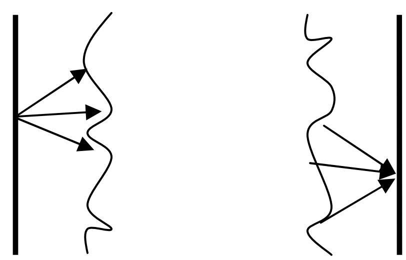


(c) (3 points)



Input plane Output plane

(d) (3 points)



Input plane Output plane

Problem 3: Gaussian Beams and Resonators

(5 points)

Suppose you have a resonator with *identical mirrors* as shown below. If we increase the radius of curvature of both mirrors but *don't* modify the resonator length, will the waist radius of the resonator mode increase or decrease? Why?

Problem 4: Polarization of Light

(8 points)

A plane electromagnetic wave propagates in free space along the positive z-axis. The electric field vector of the wave is given as

$$\vec{E}(z,t) = E_{0x}\cos(\omega t - kz)\vec{e}_x + E_{0y}\cos(\omega t - kz + \varphi)\vec{e}_y \quad (1)$$

with

$$\varphi = \pi/2\,,\ E_{0y} = \sqrt{3}E_{0x}$$

(a) (3 point) What is the polarization of the wave i.e. is the light linearly polarized, circularly polarized, or elliptically polarized?

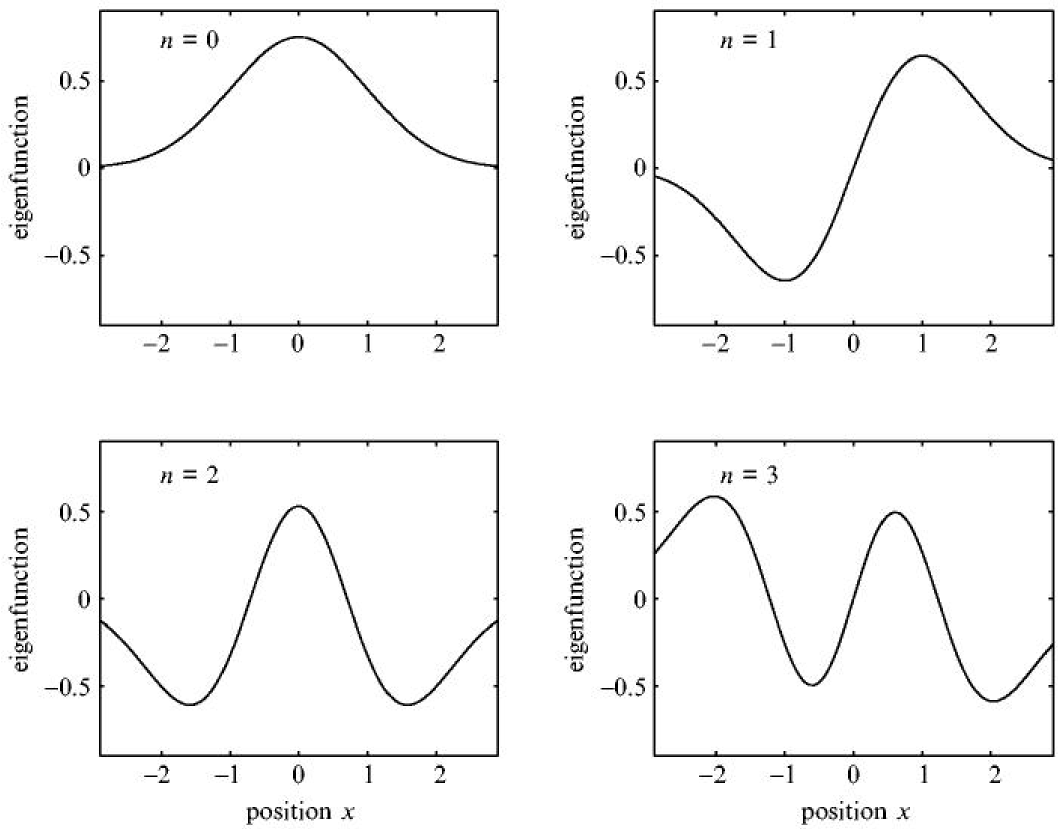
(b) (15 points) We would like to transform this wave into a circularly polarized wave using one or several half-wave and/or quarter-wave plates. Give a sequence of half-wave and quarter-wave plates and their rotation angles with respect to the *x*-axis such that the output is circularly polarized. If the answer is not unique, please give at least one combination that works.

(b) continued

Problem 5: Harmonic Oscillator

(20 points)

In class, we determined the wave functions, $\psi_n(x)$, of a one dimensional harmonic oscillator with $V(x) = \frac{1}{2}m\omega^2 x^2$. Below are the solutions for the first 4 modes: (note n indicates the state number)



Now consider a new potential defined as:

$$V(x) = \begin{cases} \infty & for \;\; x < 0 \\ \frac{1}{2}m\omega^2 x^2 & for \;\; x > 0 \end{cases}$$

This is a model potential for a spring which can only be extended but not compressed.

(a) (5 points) Sketch the eigenfunctions of the *ground* and *first excited* state of this new potential. These are the first two states of this new oscillator. We will call these normalized eigenfunctions $\phi_1(x)$ and $\phi_2(x)$.

Hints: It is not necessary to solve the Schrödinger Equation. Consider the value of ϕ_n at x = 0.

(b) (5 points) Suppose at t = 0 the state of this system is

$$\phi(x,t=0) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + \phi_2(x)\right).$$

What is the time dependant wave function, $\phi(x,t)$, for t > 0.

(c) (5 points) If we measure the energy at a time $t > 0$, what is the expected value of the energy? Express your answer in terms of $\hbar\omega$.

(d) (5 points) What is the uncertainty associated with measuring energy?

Problem 6: Operators and Commutators

(10 points)

We are given an operator b such that

$$b = \mu a + \upsilon a^{\dagger}$$

where a is the creation operator, $a^{\dagger}$ is the annihilation operator, and μ and υ are real constants related by $\mu^2 - \upsilon^2 = 1$.

(a) (5 points) Show that $\left[b, b^{\dagger}\right] = \left[a, a^{\dagger}\right]$.

(b) (5 points) If $X = \frac{1}{\sqrt{2}}\left(b^{\dagger} + b\right)$ and $P = \frac{j}{\sqrt{2}}\left(b^{\dagger} - b\right)$, are X and P Hermitian? Justify your answer.

Problem 7: Two-Level Atom

(15 points)

(a) (6 points) You are given a 2-level atom in a resonant interaction with a monochromatic, classical, electromagnetic wave. The levels of this system are separated by an energy (E_e-E_g) and initially the atom is in the ***excited*** state. Sketch the evolution of the occupation probabilities of the ground and excited states. Label your sketches appropriately and physically interpret this time evolution.

(b) (9 points) We found for the average dipole moment and inversion of an ensemble of two level atoms, the following equations of motion:

$$\dot{d} = \left(j\omega_{eg} - \frac{1}{T_2} \right) d \quad \text{and} \quad \dot{w} = -\frac{w - w_0}{T_1}.$$

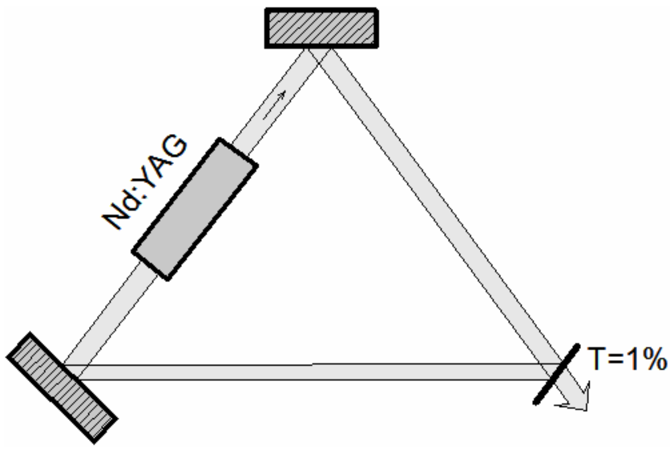
(i) What do the time constants T_1 and T_2 represent? In real systems, which value is most often smaller and why?

(ii) What does the term w_0 represent? At thermal equilibrium, is w_0 positive or negative? Justify your answer.

Problem 8: Laser

(25 points)

We will consider the ring laser shown in the figure below. It has the following parameters: stimulated emission cross section: $\sigma = 3 \cdot 10^{-19} cm^2$, upper state lifetime $\tau_L = 3\mu s$, effective mode cross sectional area $A_{eff} = 10^{-4} cm^2$, cavity length $L = 1m$, output coupling mirror transmission 1%, V is the volume of the laser mode V = $A_{eff} \cdot L$, and τ_p the photon lifetime.



The reduced rate equations for the population in the upper laser level, N_2, in the four level laser material and the photon number, N_L, in the laser mode are

$$\frac{dN_2}{dt} = -\frac{1}{\tau_L} N_2 - \frac{\sigma c}{V} N_2 \cdot N_L + R_P$$
$$\frac{dN_L}{dt} = -\frac{1}{\tau_p} N_L + \frac{\sigma c}{V} N_2 \cdot (N_L + 1) \quad (1)$$

Where R_P is the pump rate of the laser and $c = 3 \cdot 10^8 m/s$ is the speed of light which shall be for simplicity equal to the group velocity in the laser material.

(a) (5 points) Give a physical interpretation of each of the terms in equation (1).

$-\frac{1}{\tau_L} N_2$:

$-\frac{\sigma c}{V} N_2 \cdot N_L$:

R_P :

$-\frac{1}{\tau_p} N_L$:

$+\frac{\sigma c}{V} N_2$:

(b) (5 points) Determine the minimum pump rate, R_P, necessary to reach the lasing threshold. How much pump power does this pump rate correspond to, if optical pumping with photons with 532nm wavelength would be used?

(c) (5 points) Let's assume now, that the laser has in addition to the losses due to output coupling also internal losses of 0.5%. Derive an expression for the output power for the laser as a function of the laser saturation power, the small signal gain per roundtrip, the internal losses and the output coupling mirror transmission, *T*.

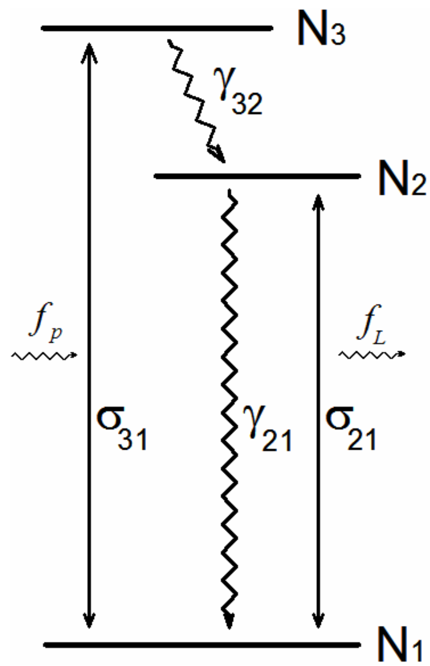
(d) (10 points) For a given small-signal gain per round-trip, what is the optimum output coupling mirror transmission *T* to achieve maximum output power.

(d) continued

Problem 9: Three-Level System

(<u>Graduate problem</u>, 25 points)

Consider an atom with three energy levels as shown in the figure. The pump at frequency $f_p = \frac{E_3 - E_1}{h}$ causes stimulated transitions between the 1st and the 3rd levels; the cross-section of this process is σ_{31}. The 3rd level exhibits a spontaneous decay rate to the 2nd level of γ_{32}. The second level is long-lived with a spontaneous decay rate of $\gamma_{21} < \gamma_{32}$. With sufficient pumping, population inversion can be achieved between levels 2 and 1 so that the light at frequency $f_L = \frac{E_2 - E_1}{h}$ will be amplified. The cross-section of stimulated transitions between levels 2 and 1 is σ_{21}.



(a) *(8 points)* In some systems of practical interest the relaxation from the 3rd to the 2nd level is **not** fast enough to ignore the population of the 3rd level. Write the rate equations for the number of atoms in levels 1, 2, and 3, N_1, N_2, and N_3. The pump intensity is I_p. Assume that the intensity at frequency f_L is <u>*very small*</u> so that it does not change the population of the energy levels. The total number of atoms is N.

(b) *(8 points)* Find the steady-state population of the second level N_2.

(c) *(2 points)* What is the population of the 2nd level for very large pump intensity?

For the remaining two questions, consider the case of very fast relaxation from the 3rd level, $\gamma_{32} >> \gamma_{21}$.

(d) *(2 points)* What is the population of the 2nd level for very large pump intensity? Explain the physical difference between this answer and that of part (c).

(e) *(5 points)* What is the lowest pump intensity needed to achieve amplification?

Equation Sheets

Maxwell's Equations	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$	
	$\nabla \cdot \vec{D} = \rho$	$\nabla \cdot \vec{B} = 0$	
Material Equations	$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon \vec{E}$	$\vec{P} = \varepsilon_0 \chi_e \vec{E}$	$\varepsilon = \varepsilon_0 (1 + \chi_e)$
	$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu \vec{H}$	$\vec{M} = \chi_m \vec{H}$	$\mu = \mu_0 (1 + \chi_m)$
Index of Refraction	$n^2 = 1 + \chi$,	for $\chi << 1$: $n \approx 1 + \chi / 2$	
Poynting Vector	$\vec{S} = \vec{E} \times \vec{H}$	$\vec{T} = \frac{1}{2} \underline{\vec{E}} \times \underline{\vec{H}}^*$	
Energy density	$w_e = \frac{1}{2} \varepsilon \vec{E}^2$	$w_m = \frac{1}{2} \mu \vec{H}^2$	$w = w_e + w_m$
Snell's Law	$n_1 \sin \theta_1 = n_2 \sin \theta_2$		
Brewster's Angle	$\tan \theta_B = \frac{n_2}{n_1}$		
Reflectivity	$r^{TE} = \frac{Z_2^{TE} - Z_1^{TE}}{Z_1^{TE} + Z_2^{TE}}$	$r^{TM} = \frac{Z_1^{TM} - Z_2^{TM}}{Z_1^{TM} + Z_2^{TM}}$	
	$Z_{1/2}^{TE} = \sqrt{\frac{\mu_{1/2}}{\varepsilon_{1/2}}} \frac{1}{\cos \theta_{1/2}}$	$Z_{1/2}^{TM} = \sqrt{\frac{\mu_{1/2}}{\varepsilon_{1/2}}} \cos \theta_{1/2}$	
	$r^{TE} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2}$	$r^{TM} = \frac{\frac{n_2}{\cos \theta_2} - \frac{n_1}{\cos \theta_1}}{\frac{n_2}{\cos \theta_2} + \frac{n_1}{\cos \theta_1}}$	
Transmitivity	$t^{TE} = \frac{2 Z_2^{TE}}{Z_1^{TE} + Z_2^{TE}}$	$t^{TM} = \frac{2 Z_1^{TM}}{Z_1^{TM} + Z_2^{TM}}$	
	$t^{TE} = \frac{2 n_1 \cos(\theta_1)}{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)}$	$t^{TM} = \frac{2 \frac{n_2}{\cos(\theta_2)}}{\frac{n_2}{\cos(\theta_2)} + \frac{n_1}{\cos(\theta_1)}}$	
Power Refl. Coef.	$R^{TE} = \left| r^{TE} \right|^2$	$R^{TM} = \left| r^{TM} \right|^2$	
Power Transm. Coef.	$T^{TE} = \left| t^{TE} \right|^2 \frac{Z_1^{TE}}{Z_2^{TE}} = \frac{4 Z_1^{TE} Z_2^{TE}}{\left| Z_1^{TE} + Z_2^{TE} \right|^2}$	$T^{TM} = \left| t^{TM} \right|^2 \frac{Z_2^{TM}}{Z_1^{TM}} = \frac{4 Z_1^{TM} Z_2^{TM}}{\left| Z_1^{TM} + Z_2^{TM} \right|^2}$	

Pulse Dispersion $$\frac{\partial A(z,t')}{\partial z} = j\frac{k''}{2}\frac{\partial^2 A(z,t')}{\partial t'^2}$$

Gaussian Pulse $$\tau(L) = \tau\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2} \qquad \tau_{FWHM} = 2\sqrt{\ln 2}\,\tau$$

Fabry Perot $$|S_{21}|^2 = \frac{(1-R)^2}{(1-R)^2 + 4R\sin^2(\phi/2)} \quad \text{where } \phi = 2kL,\ k = \frac{2\pi f}{c_0}n$$

$$f_m = m\frac{c_0}{2nL} \qquad FSR = \frac{c_0}{2nL}$$

$$F = \frac{FSR}{\Delta f_{FWHM}} \qquad F = \frac{\pi\sqrt{R}}{1-R} \approx \frac{\pi}{1-R}$$

Beam Splitter S-matrix $S = \begin{pmatrix} r & jt \\ jt & r \end{pmatrix}$, with $r^2 + t^2 = 1$

Gaussian Beams $$E(r,z) = \frac{\sqrt{2P}}{\sqrt{\pi}w(z)}\exp\left[-\frac{r^2}{w^2(z)} - jk_0\frac{r^2}{2R(z)} + j\zeta(z)\right]$$

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \qquad R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right]$$

$$\zeta(z) = \arctan\left(\frac{z}{z_R}\right) \qquad I(r,z) = I_0\frac{w_0^2}{w(z)^2}\exp\left[-\frac{2r^2}{w^2(z)}\right]$$

$$z_R = \frac{k\,w_0^2}{2} = \frac{\pi\,w_0^2}{\lambda} \qquad \theta = \frac{\lambda}{\pi\,w_0}$$

q-parameter $$q(z) = z + jz_R \qquad \frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi\,w^2(z)}$$

$$E(r,z) = \frac{1}{q(z)}\exp\left[-jk_0\frac{r^2}{2q(z)}\right]$$

Resonator Stability $$0 \le g_1g_2 \le 1 \quad g_1 = 1-\frac{L}{R_1} \qquad g_2 = 1-\frac{L}{R_2}$$

Schrodinger's Equation $$j\hbar\frac{d\Psi(r,t)}{dt} = -\frac{\hbar^2}{2m}\Delta\Psi(r,t) + V(r)\Psi(r,t)$$

Harmonic Oscillator $$E = \hbar\omega\left(n+\frac{1}{2}\right)$$

Heisenberg's Uncertainty Principle $\Delta x \Delta p \geq \frac{\hbar}{2}$

de Broglie's Formula $p = \hbar k$

Einstein's Energy/Frequency Relation $E = \hbar\omega$

Wien's Law $w(f) = \frac{8\pi h f^3}{c^3} e^{-hf/kT}$

Wien Displacement Law $\lambda_{\max} = \frac{hc}{4.965kT}$

Rayleigh-Jeans Law $w(f) = \frac{8\pi}{c^3} f^2 kT$

Planck's Law $w(f) = \frac{8\pi f^2}{c^3} \frac{hf}{\exp\frac{hf}{kT} - 1}$

Einstein's A and B Coefficients $A_{21} = \frac{8\pi f_{21}^3}{c^3} B_{12} \qquad B_{21} = B_{12}$

Waveguide Coupling $P_1(z) = P_1(0)\left(\cos^2 \gamma z + \left(\frac{\Delta\beta}{\gamma}\right)^2 \sin^2 \gamma z\right)$

$$P_2(z) = P_1(0)\frac{|\kappa_{21}|^2}{\gamma^2}\sin^2(\gamma z) \qquad \gamma = \sqrt{\Delta\beta^2 + |\kappa_{12}|^2} \qquad \Delta\beta = \frac{\beta_1 - \beta_2}{2} \quad \beta_0 = \frac{\beta_1 + \beta_2}{2}$$

Polarization, Retardation Plate

$$W = \begin{pmatrix} e^{-j\Gamma/2}\cos^2\psi + e^{j\Gamma/2}\sin^2\psi & -j\sin\left(\frac{\Gamma}{2}\right)\sin(2\psi) \\ -j\sin\left(\frac{\Gamma}{2}\right)\sin(2\psi) & e^{-j\Gamma/2}\sin^2\psi + e^{j\Gamma/2}\cos^2\psi \end{pmatrix}$$

$$\text{if } \Gamma = \pi: \qquad W = -j\begin{pmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{pmatrix}$$

$$\text{if } \Gamma = \frac{\pi}{2}: \qquad W = \begin{pmatrix} \frac{1}{\sqrt{2}}[1 - j\cos(2\psi)] & -j\frac{1}{\sqrt{2}}\sin(2\psi) \\ -j\frac{1}{\sqrt{2}}\sin(2\psi) & \frac{1}{\sqrt{2}}[1 + j\cos(2\psi)] \end{pmatrix}$$

Rabi Frequency $\Omega_r = \frac{\overrightarrow{M} \cdot \overrightarrow{e_p^*}}{2\hbar} \underline{E_0}$

Annihilation and Creation Operators $\left[a, a^{\dagger}\right] = 1$

Bloch Equations

$$\dot{d} = -\left(\frac{1}{T_2} - j\omega_{eg}\right)d + j\Omega_r^* e^{j\omega t}\omega$$

$$\dot{\omega} = -\frac{\omega - \omega_0}{T_1} + 2j\Omega_r e^{-j\omega t}d - 2j\Omega_r^* e^{j\omega t}d^*$$

Trigonometric Identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$$

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$$

$$\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha + \beta) + \frac{1}{2}\sin(\alpha - \beta)$$

$$\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha + \beta) - \frac{1}{2}\sin(\alpha - \beta)$$

Helpful Integrals

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{\infty} xe^{-ax^2}dx = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\int \alpha\sin\alpha \, d\alpha = \sin\alpha - \alpha\cos\alpha + C$$

$$\int \alpha\cos\alpha \, d\alpha = \cos\alpha - \alpha\sin\alpha + C$$

Constants

$\varepsilon_o = 8.85 \times 10^{-12}$ C^2/Nm^2	permittivity of free space
$\mu_o = 4\pi \times 10^{-7}$ N/A^2	permeability of free space
m = 9.11×10^{-31} kg	mass of an electron
e = 1.60×10^{-19} C	charge of an electron
k = 1.380650×10^{-23} J/K	Boltzmann's constant
$c_0 = 2.997925 \times 10^8$ m/s	speed of light in free space