0.1 Review

general graphs: adjacent vertices:

- lemma: for adjcaent (u, v), $h_{uv} + h_{vu} \leq 2m$
- proof: new markov chain on *edge traversed following vertex MC*
  - transition matrix is *doubly stochastic:* column sums are 1 (exactly $d(v)$ edges can transit to edge (v, w), each does so with probability $1/d(v)$)
  - In homework, show such matrices have *uniform* stationary distribution.
  - Deduce $\pi_e = 1/2m$. Thus $h_{ee} = 2m$.
- So consider suppose original chain on vertex v.
  - suppose arrived via (u, v)
  - expected to traverse (u, v) again in $2m$ steps
  - at this point will have commuted u to v and back.
  - so conditioning on arrival method, commute time $2m$ (thanks to memorylessness)

General graph cover time:

- theorem: cover time $O(mn)$
- proof: find a spanning tree
- consider a dfs of tree-crosses each edge once in each direction, gives order $v_1, \ldots, v_{2n-1}$
- time for the vertices to be visited in this order is upper bounded by commute time
- but vertices adjacent, so commute times $O(m)$
- total time $O(mn)$
- tight for lollipop, loose for line.

Applications

Testing graph connectivity in logspace.

- Deterministic algorithm (matrix squaring) gives $\log^2 n$ space
- Smarter algorithms gives $\log^{4/3} n$ space
- $\log n$ open
- Randomized logspace achieves one-sided error

universal traversal sequences.

- Define labelled graph
- UTS covers any labelled graph
- deterministic construction known for cycle only
- we showed cover time $O(n^3)$
- so probability takes more than $2n^3$ to cover is $1/2$
- repeat k times. Prob fail $1/2^k$
- How many graphs? $(nd)^{O(nd)}$
- So set $k = O(nd \log nd)$
- probabilistic method

Nisan $n^{O(\log n)}$ via pseudorandom generator that fools logspace machines.

Markov Chains for Sampling

Sampling:

- Given complex state space
- Want to sample from it
- Use some Markov Chain
- Run for a long time
- end up "near" stationary distribution
- Reduces sampling to local moves (easier)
- no need for global description of state space
- Allows sample from exponential state space

Formalize: what is "near" and "long time"?

- Stationary distribution π
- arbitrary distribution q
- **relative pointwise distance (r.p.d.)** $\max_j |q_j - \pi_j|/\pi_j$
- Intuitively close.
- Formally, suppose r.p.d. δ.
- Then $(1-\delta)\pi \le q$

- So can express distribution q as "with probability $1 - \delta$, sample from π. Else, do something wierd.
- So if δ small, "as if" sampling from π each time.
- If δ poly small, can do poly samples without goof
- Gives "almost stationary" sample from Markov Chain
- Mixing Time: time to reduce r.p.d to some ϵ

Eigenvalues

Method 1 for mixing time: Eigenvalues.

- Consider transition matrix P.
- Eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$
- Corresponding Eigenvectors $e_1, \ldots, e_n$.
- Any vector q can be written as $\sum a_i e_i$
- Then $qP = \sum a_i \lambda_i e_i$
- and $qP^k = \sum a_i \lambda_i^k e_i$
- so sufficient to understand eigenvalues and vectors.
- Is any $|\lambda_i| > 1$?
  - If so, $e_i P = \lambda_i P$
  - let M be max entry of e_i (in absolute value)
  - if $\lambda_i > 1$, then some $e_i P$ entry is $\lambda_i M > M$
  - any entry of $e_i P$ is a convex combo of values at most M, so max value M, contradiction.
  - Deduce: all eigenvalues of stochastic matrix at most 1.
- How many $\lambda_i = 1$?
  - Stationary distribution ($e_1 = \pi$)
  - if any others, could add a little bit of it to e_1, get second stationary distribution
  - What about -1? Only if periodic.
- so all other coordinates of eigenvalue decomposition **decay** as λ_i^k.
- So if can show other λ_i small, converge to stationary distribution fast.
- In particular, if $\lambda_2 < 1 - 1/poly$, get polynomial mixing time

Expanders:

Definition

- bipartite
- n vertices, regular degree d
- $|\Gamma(S)| \geq (1 + c(1 - 2|S|/n))|S|$

factor c more neighbors, at least until S near $n/2$.
Take random walk on (n, d, c) expander with constant c

- add self loops (with probability 1/2 to deal with periodicity.
- uniform stationary distribution
- lemma: second eigenvalue $1 - O(1/d)$

$$\lambda_2 \leq 1 - \frac{c^2}{d(2048 + 4c^2)}$$

- Intuition on convergence: because neighborhoods grow, position becomes unpredictable very fast.
- proof: messy math

Deduce: mixing time in expander is $O(\log n)$ to get ϵ r.p.d. (since $\pi_i = 1/n$)
Converse theorem: if $\lambda_2 \leq 1 - \epsilon$, get expander with

$$c \geq 4(\epsilon - \epsilon^2)$$

Walks that mix fast are on expanders.
Gabber-Galil expanders:

- Do expanders exist? Yes! proof: probabilistic method.
- But in this case, can do better deterministically.
  - Gabber Galil expanders.
  - Let $n = 2m^2$. Vertices are (x, y) where $x, y \in Z_m$ (one set per side)
  - 5 neighbors: $(x, y), (x, x + y), (x, x + y + 1), (x + y, y), (x + y + 1, y)$ (add mod m)
  - or 7 neighbors of similar form.
- Theorem: this $d = 5$ graph has $c = (2 - \sqrt{3})/4$, degree 7 has twice the expansion.
- in other words, c and d are constant.
- meaning $\lambda_2 = 1 - \epsilon$ for some **constant** ϵ
- So random walks on this expander mix *very* fast: for polynomially small r.p.d., $O(\log n)$ steps of random walk suffice.
- Note also that n can be huge, since only need to store one vertex ($O(\log n)$ bits).