## Complexity.

What is a rand. alg?
What is an alg?

- Turing Machines. RAM with large ints. log-cost RAM as TM.
- language as decision problem (vs optimization problems) "graphs with small min-cut." algos accept/reject
- complexity class as set of languages
- $P$. polynomial time in input size
- $NP$ as $P$ with good advice string. **witnesses**
- polytime reductions. hardness, completeness.

Randomized algorithms have advice string, but it is **random**

- measure probs over space of advice strings
- equivalence to fliping unbiased random bits

$ZPP$ (zero error probabilistic polytime)

- Polynomial expected time
- $A(x)$ accepts iff $x \in L$.
- Las Vegas algorithms

$RP$ (randomized polytime) (MC with one-sided error).

- polytime (always)
- $x \notin L \Rightarrow$ rejects (always).
- $x \in L \Rightarrow$ accepts with probability $> 1/2$.
- Monte Carlo algorithm
- one sided error
- precise numbers unimportant: **amplification**.
- min-cut example
- $coRP$.
- What if **NOT** worst case polytime? stop when passes time bound and accept.
- $ZPP = RP \cap coRP$

$PP$ (probabilistic polytime) (two-sided MC)

- Worst case polytime (can force)
- $x \in L \Rightarrow$ accepts prob $> 1/2$
- $x \notin L \Rightarrow$ accepts prob $< 1/2$
- weakness: $NP \subseteq PP$

$BPP$ (bounded probabilistic polytime)

- worst case polytime (can force)
- $x \in L \Rightarrow$ accepts prob $> 3/4$
- $x \notin L \Rightarrow$ accepts prob $< 1/4$
- precise numbers unimportant.

Clearly $P \subseteq RP \subseteq NP$. Open questions:

- $RP = coRP$? (equiv $RP = ZPP$)
- $BPP \subseteq NP$?

## Tree evaluation.

Moving LOE through a (linear) recurrence.

- define. algo cost is number of leaves. $n = 2^h$
- NOR model

deterministic model: must examine all leaves. time $2^h = 4^{h/2} = n$

- by induction: on any tree of height $h$, as questions are asked, can answer such that root is not determined until all leaves checked.
- Note: bad instance being constructed on the fly as algorithm runs.
- But, since algorithm deterministic, bad instance can be built in advance by simulating algorithm.

nondeterministic/checking

- $W(0) = L(0) = 1$
- winning position can guess move. $W(h) = L(h-1)$
- losing must check both. $L(h) = 2W(h-1)$
- follows $W(h) = 2 * W(h-2) = 2^{h/2} = n^{1/2}$

randomized–guess which leaf wins.

- $W(0) = 1$
- $W(T)$ is a random variable
  - If $T$ is winning time it takes to verify $T$ is a win. Undefined if $T$ is losing.
  - Ditto $L(T)$.
  - Expectation is over random choices of algorithm; NOT over trees.
  - Different trees have different expectations
- $W(h) =$ max over all height-$h$ winning trees of $E[W(T)]$
- $L(h) =$ same for losing trees.
- Consider any losing height-$h$ tree
  - both children are winning
  - must eval both.
  - each takes at most $W(h-1)$ in expectation
  - Thus (by linearity of expectation) we take at most $2W(h-1)$
  - Deduce $L(h) \leq 2W(h-1)$.
- Consider any winning height-$h$ tree
  - Possibly both children are losing. If so, we stop after evaling the first child we pick. Total time $L(h-1)$.
  - If exactly one child losing, two cases:
    * if first choice is winning, eval it and stop: time at most $L(h-1)$.
    * if first choice is losing, eval both children: $L(h-1) + W(h-1)$.
    * Conjecture: $W(h-1) \leq L(h-1)$
    * Then time $\leq 2L(h-1)$.
  - Each case 1/2 the time. Thus, expected time $\leq (3/2)L(h-1)$.
  - Deduce $W(h) \leq (3/2)L(h-1) \leq (3/2)2W(h-2) = 3W(h-2)$
  - So $W(h) \leq 3^{h/2} = n^{\log_4 3} = n^{0.793}$
  - Go back and confirm assumption that $W(h) \leq L(h)$.