

6.856 — Randomized Algorithms

David Karger

Handout #15, October 23, 2002 — Homework 8, due 10/30

M. R. refers to this text:

Motwani, Rajeez, and Prabhakar Raghavan. *Randomized Algorithms*. Cambridge: Cambridge University Press, 1995.

1. Suppose you are given a graph whose edge lengths are all integers in the range from 0 to B . Suppose also that you are given the all-pairs distance matrix for this graph (it can be constructed by a variant of Seidel's distance algorithm). Prove that you can identify the (successor matrix representation of the) shortest paths in $O(B^2 MM(n) \log^2 n)$ time, where $MM(n)$ is the time to multiply $n \times n$ matrices.
2. **M. R. 12.16** Consider the problem of finding a *minimum weight* perfect matching in a graph whose edges are given integer weights of magnitude bounded by a polynomial in the number of vertices n . Note that it is not possible to apply the Isolating Lemma directly to this case since the random weights chosen there would conflict with the input weights. Explain how you would devise an \mathcal{RNC} algorithm for this problem. **Hint:** start by scaling up the input edge weights by a large polynomial factor. Apply random perturbations to the scaled weights and prove a variant of the Isolating Lemma for this situation.

The parallel complexity of the version where the edge weights can have a polynomial number of *bits* has not yet been resolved. Note that arithmetic operations on such weights are still tractable. But do you see why the \mathcal{RNC} algorithm you developed above does not work in this case?

3. **MR 12.12.** Consider the following alternative parallel algorithm for maximal independent set. In a phase, a (non maximal) independent set S is output, and all of its neighbors are deleted. To find S , assign to each vertex a uniformly deistributed random weight from the range $\{1, \dots, n^4\}$. Mark all vertices, then in parallel unmark the larger-weight endpoint of each edge. The set of vertices that remain marked is S . Continue until the graph is empty.
 - (a) Argue that the set of vertices output over all phases is a maximal independent set.
 - (b) Assuming that v has degree d during a phase, what is the probability that vertex v stays marked?
 - (c) Show that this approach yields an \mathcal{RNC} algorithm for maximal independent set.
4. **MR 12.24.** Consider an undirected graph G . A assignment of lengths to the edges determines shortest paths between all vertices. An assignment is *good* if

- All shortest paths are unique, and
- All pairwise distances in the graph are distinct

Find the smallest W you can for which there *exists* a good assignment of weights all drawn from the range $\{0, \dots, W\}$.

5. **Optional.** Devise an \mathcal{RNC} algorithm for finding a *maximum matching* (i.e., most possible edges) in a graph that may not have a perfect matching. **Hint:** use the min-weight perfect matching algorithm above as a “black box” by making nonexistent edges very expensive.
6. **Optional.** In the *exact matching* problem, a bipartite graph is given with a subset of the edges colored red, along with an integer k . The goal is to find a perfect matching with exactly k red edges. Devise an \mathcal{RNC} algorithm for this problem using a (non-trivial) application of the Isolating Lemma. Note that this problem is not known to be solvable in \mathcal{P} .