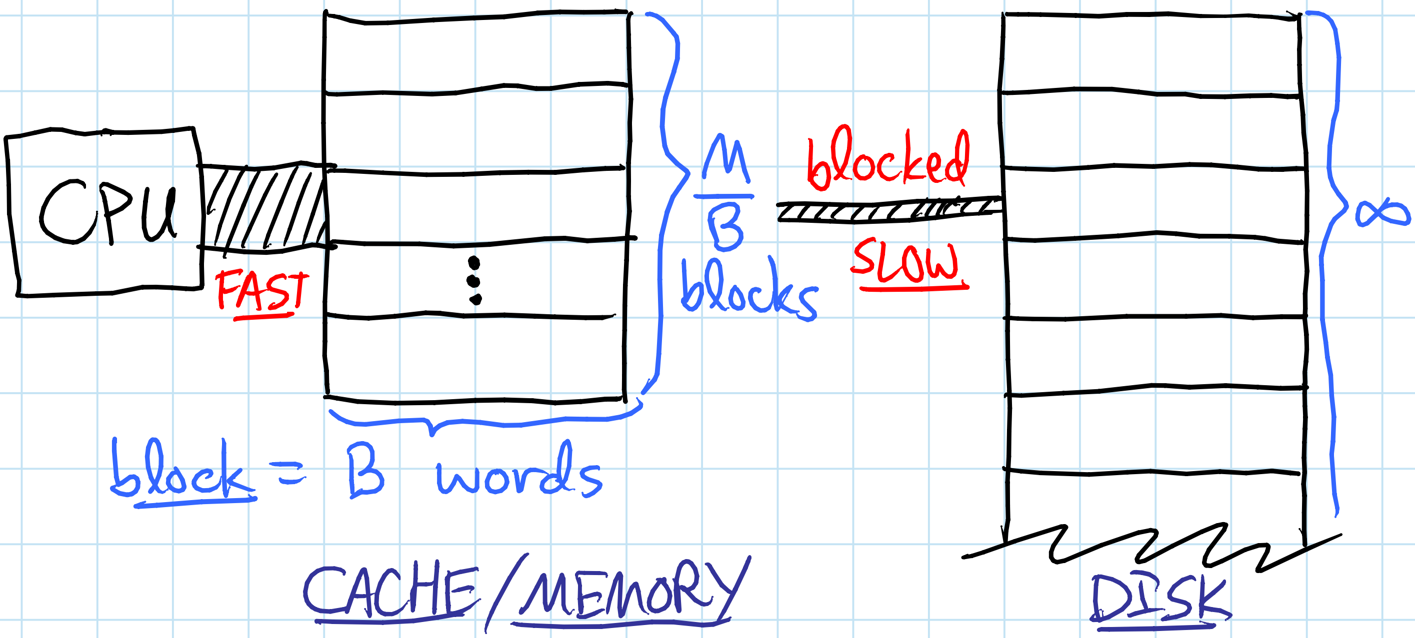
6.851	Lecture 7	Mar. 8, 2012

TODAY: Memory Hierarchies I (of 3)
- external-memory model
- cache-oblivious model
- cache-oblivious B-trees

External memory / I/O / Disk Access Model:

[Aggarwal & Vitter – CACM 1988]

two-level memory hierarchy

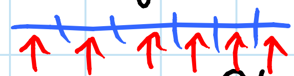


- focus on # memory transfers:
  blocks read/written between cache & disk
  - $\leq$ RAM running time
  - $\geq \dfrac{\text{cell-probe LB}}{B}$
  - when can we save this factor of $\geq B$?

Basic results in external memory:

⓪ Scanning: $O(\lceil \frac{N}{B} \rceil)$ to read/write N words in order

① Search trees:

- B-trees with branching factor $\Theta(B)$ support insert, delete, predecessor search in $O(\log_{B+1} N)$ memory transfers
  (& $O(\lg N)$ time, with care, in comparison model)
- $\Omega(\log_{B+1} N)$ for search in comparison model:
  - where query fits among N items requires $\lg(N+1)$ bits of information 
  - each block read reveals where query fits among B items $\Rightarrow \leq \lg(B+1)$ bits of info.

  $\Rightarrow$ need $\geq \dfrac{\lg(N+1)}{\lg(B+1)}$ memory transfers
- also optimal in "block-probe model" if $B \geq w$
  [Pătrașcu & Thorup — see L11]

② Sorting: $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ memory transfers
↳ $B\times$ faster than B-tree sort!
Ω(ditto) in comparison model

③ Permuting: $O(\min\{N, \frac{N}{B} \log_{M/B} \frac{N}{B}\})$
physical execution
Ω(ditto) in indivisible model
↳ can't pack pieces of input words in words

④ Buffer tree: $O(\frac{1}{B} \log_{M/B} \frac{N}{B})$ amortized mem. transf. for delayed queries & batched updates & $O(\emptyset)$ delete-min ($\Rightarrow$ priority queues)

Cache-oblivious model: [Frigo, Leiserson, Prokop, Ramachandran — FOCS 1999; Prokop — MEng 1999] 6.046

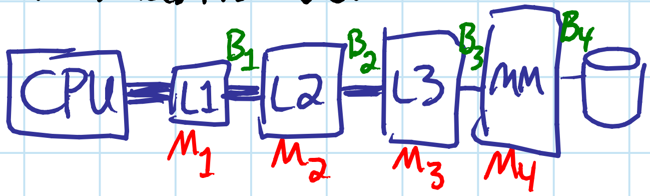
- like external-memory model
- but algorithm doesn't know B or M (!)

⇒ must work for all B & M

- automatic block transfers triggered by word access with offline optimal block replacement
  - FIFO, LRU, or any conservative replacement is 2-competitive given cache of 2× size (resource augmentation)
  - dropping $M \searrow M/2$ doesn't affect typical bounds e.g. sorting bound

Cool:

- clean model: algorithm just like RAM
- adapts to changing B (disk tracks & cache) & M (competing processes)
  - OPEN: formalize this
- adapts to all levels of multilevel memory hierarchy:



- often possible!

Basic cache-oblivious results:

Ø Scanning: same (algorithm & bound)

* 1 Search trees: insert, delete, & search in $O(\log_{B+1} N)$ memory transfers — TODAY & L21

[Bender, Demaine, Farach-Colton — FOCS 2000/SICOMP 2005]
[Bender, Duan, Iacono, Wu — SODA 2002/J.Alg. 2004]
[Brodal, Fagerberg, Jacob — SODA 2002]

— best constant is lg e, not 1
[Bender, Brodal, Fagerberg, Ge, He, Hu, Iacono, López-Ortiz — FOCS 2003]

2 Sorting: $O(\frac{N}{B} \log_{M/B} \frac{N}{B})$ memory transfers
[Frigo et al. 1999; Brodal & Fagerberg — ICALP 2002]

— uses tall-cache assumption: $M = \Omega(B^{1+\varepsilon})$

— impossible otherwise [Brodal & Fagerberg — STOC 2003]

3 Permuting: min impossible [Brodal & Fagerberg — same]

* 4 Priority queue: $O(\frac{1}{B} \log_{M/B} \frac{1}{B})$ amortized mem. transf.

— uses tall-cache assumption — L21

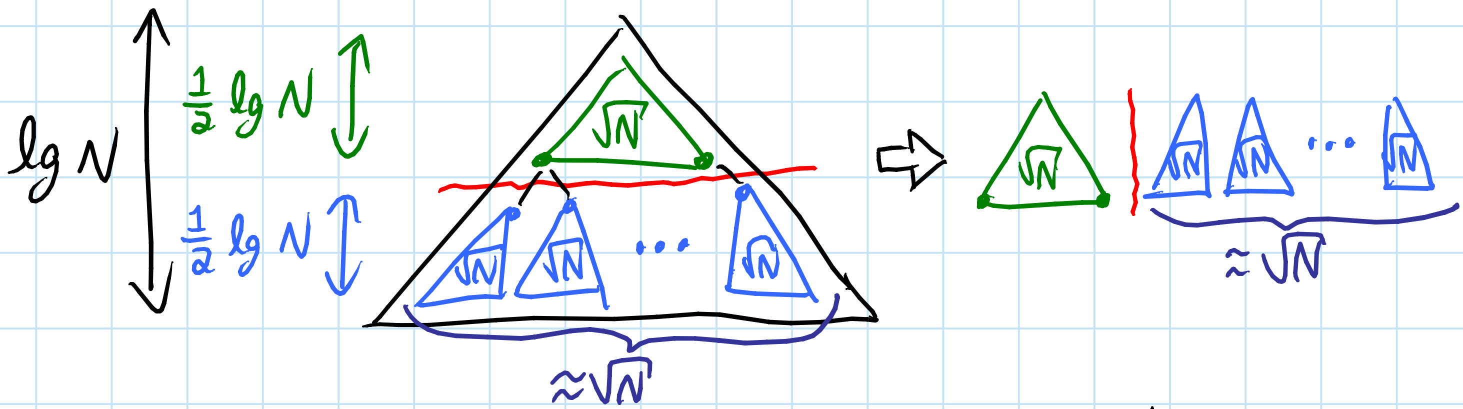
[Arge, Bender, Demaine, Holland-Minkley, Munro — STOC 2002/SICOMP 2007; Brodal & Fagerberg — ISAAC 2002]

Cache-oblivious static search trees:

(binary search) [Prokop–MEng 1999]

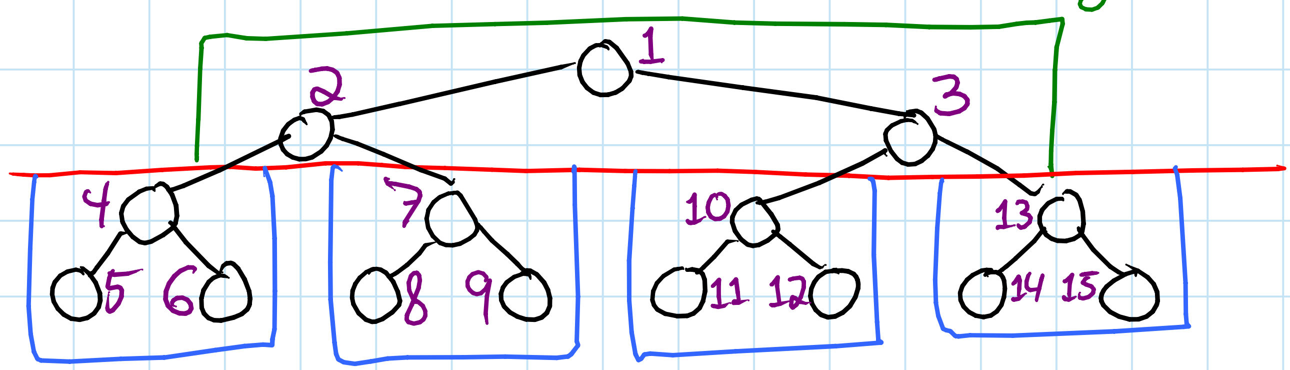
- store N elements in N-node complete BST
- carve tree at middle level of edges

$\Rightarrow$ one top piece, $\approx\sqrt{N}$ bottom pieces, each size $\approx\sqrt{N}$



- recursively lay out pieces & concatenate:

(in any order)



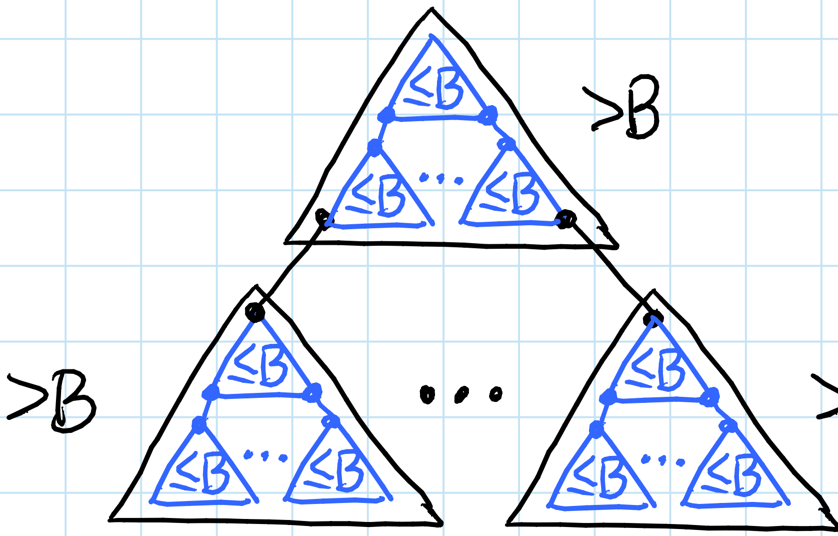
$\Rightarrow$ order to store nodes

"van Emde Boas layout"

- generalizes to [Bender, Demaine, Farach-Colton 2000]
  - height not a power of 2
  - node degrees ≥ 2 & $O(1)$

Analysis:

- level of detail (refinement) straddling B:



- cutting height in half until piece size $\le B$
  - $\Rightarrow$ height of piece between $\frac{1}{2} \lg B$ & $\lg B$ (sloppy)
    ($\Rightarrow$ size between $\sqrt{B}$ & B)
  - $\Rightarrow$ # pieces along root-to-leaf path $\le \dfrac{\lg N}{\frac{1}{2}\lg B} = 2 \log_B N$
- each piece stores $\le B$ elements consecutively
  - $\Rightarrow$ occupies ≤ 2 blocks (depending on alignment)
  - $\Rightarrow$ #memory transfers $\le 4 \log_B N$ (assuming $M \ge 2B$)
    (really should be $B+1$)

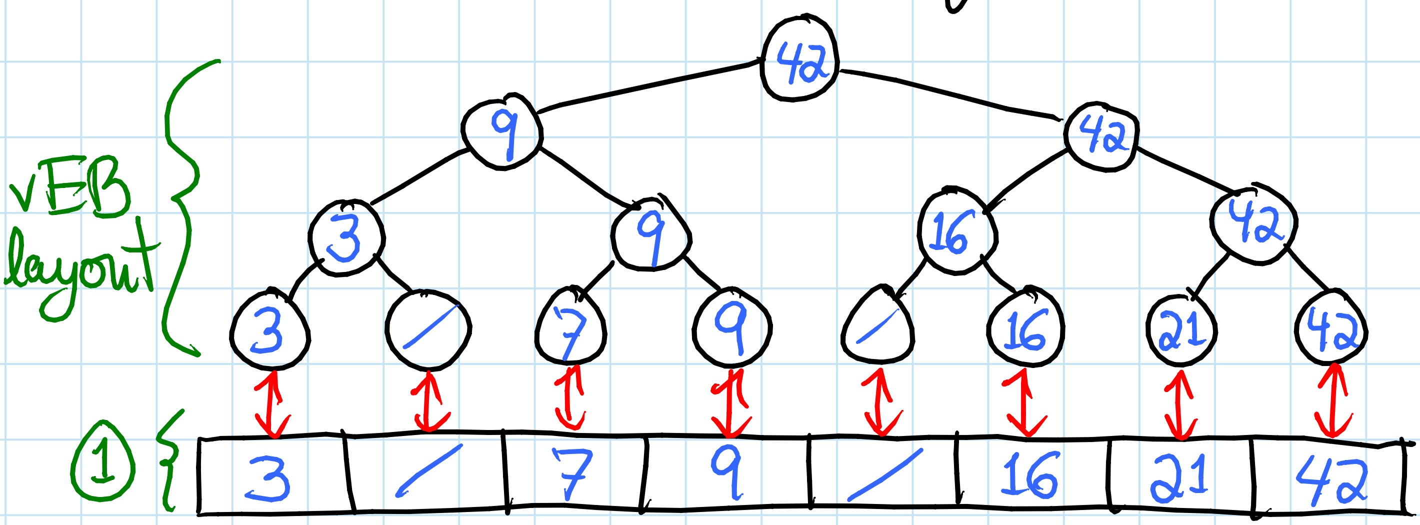
Improvements: [BBFGHHIL 2003]

1. randomize starting location (w.r.t. block)
   $\Rightarrow$ expected cost $\le \left(2 + \frac{3}{\sqrt{B}}\right) \log_B N$
2. split height into $\frac{1}{2} - \varepsilon : \frac{1}{2} + \varepsilon$ ratio
   $\Rightarrow$ expected cost $\le (\lg e + \underbrace{o(1)}_{= O(\lg \lg B / \lg B)}) \log_B N$

Cache-oblivious B-trees as in [Bender, Duan, Iacono, Wu]

① ordered file maintenance: (to do in L8)
store N elements in specified order
in an array of size $O(N)$ with $O(1)$ gaps
— updates: insert element between two given
delete element
by re-arranging array interval of $O(\lg^2 N)$ am.

② build static search tree on top:
each node stores max key in subtree (if any)



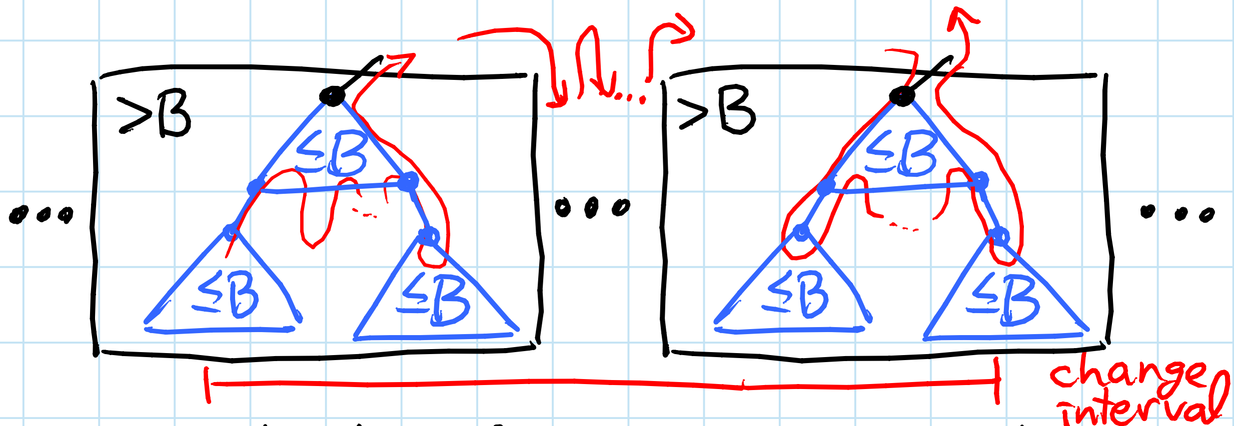
③ operations: at most double →

— binary search via left child's key
— insert(x) finds predecessor & successor,
inserts there in ordered file,
& updates leaves & max's up tree
via postorder traversal
— delete similar

④ update analysis:

if K cells change in ordered file then update tree in $O(\frac{K}{B} + \log_B N)$ mem. tr.

- look at level of detail straddling B
- look at bottom two levels:



- within chunk of $>B$, jumping between ≤ 2 pieces of $\le B$ (assume $M \ge 2B$)

$\Rightarrow O(\text{chunk}/B)$ memory transfers in chunk
portion in update interval +3 maybe (first, last, & root)

$\Rightarrow O(\frac{K}{B})$ memory transfers in bottom 2 levels

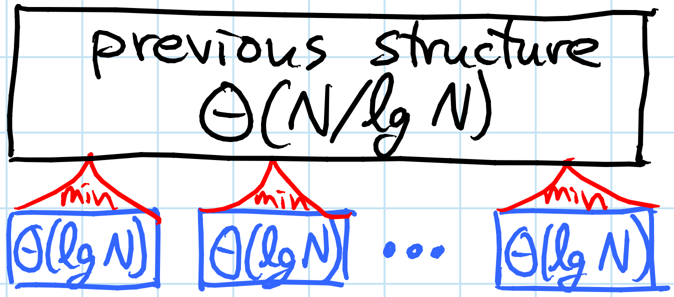
- updated nodes above these two levels:
  - subtree of $\le \frac{K}{B}$ chunk roots up to their LCA: costs $O(\frac{K}{B})$
  - path from LCA to root of tree: costs $O(\log_B N)$ as above

$\Rightarrow O(\frac{K}{B} + \log_B N)$ total memory transfers

So far: search in $O(\log_B N)$
update in $O(\log_B N + \frac{\lg^2 N}{B})$ amortized
bad if $B = o(\lg N \lg\lg N)$

⑤ <u>indirection</u>:

- cluster elements into $\Theta(\frac{N}{\lg N})$ groups, each of size $\Theta(\lg N)$
- use previous structure on min's of clusters



- update cluster by complete rewrite
  $\Rightarrow O(\frac{\lg N}{B})$ memory transfers
- split/merge clusters as necessary to keep between 25% & 100% full
  $\Rightarrow \Omega(\lg N)$ updates to charge to

$\Rightarrow O(\frac{\lg^2 N}{B})$ update cost in top structure only "every" $\Omega(\lg N)$ actual updates

$\Rightarrow$ amortized update cost $O(\frac{\lg N}{B})$ (plus search cost)

<u>Finally</u>: $O(\log_B N)$ insert, delete, predecessor, successor just like B-trees in external mem. (known B)