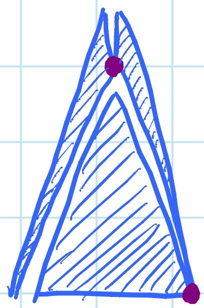
6.849	Lecture 12	Oct. 19, 2010

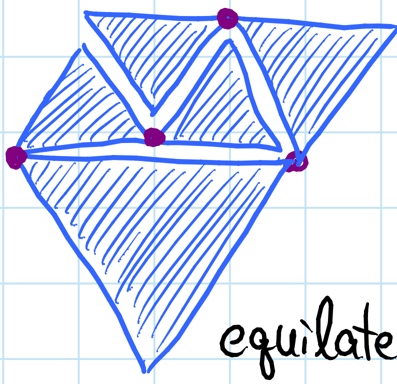
Locked & unlocked chains of planar shapes:

rigid objects in place of bars [Connelly, Demaine, Demaine, Fekete, Langerman, Mitchell, Ribó, Rote ~~2006~~ 2010]

- simple locked examples: [M. Demaine 1998]



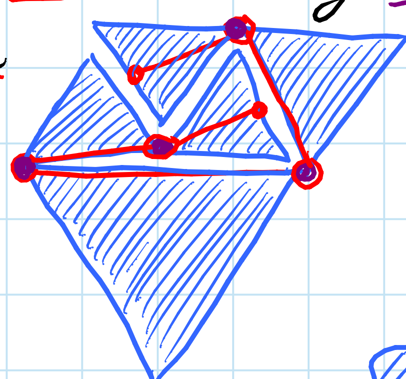
3 triangles



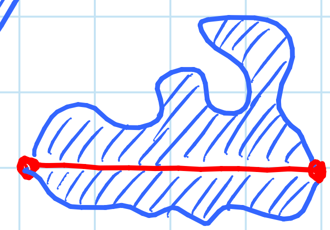
equilateral Δs

Adorned chain: view shapes as adornments attached to bar connecting hinges

- underlying chain linkage
- some flexibility in first & last shape

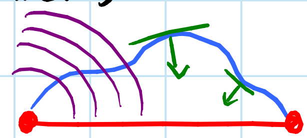


Adornment = shape + base

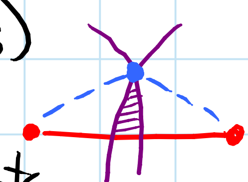


- shape = simply connected compact planar region
- base = line segment connecting 2 boundary pts.
- require base to be contained in shape

Slender adornment = walking along boundary from one base endpoint to the other monotonically increases distance to former (& decreases distance to latter)



= all inward normals hit the base (for piecewise-differentiable shapes)

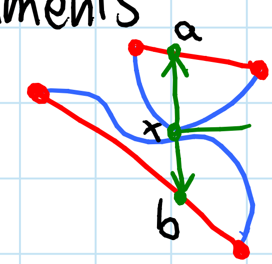


= (possibly infinite) union of half-lenses: intersection of disks centered at base endpts. & halfplane on one side of base

($\Rightarrow$ can define slender hull = union of half-lenses thru every point in adornment)

Slender $\Rightarrow$ not locked: expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Intuition: consider two touching adornments

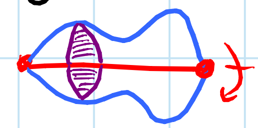


- draw collinear inward normals from touching point x
- resulting points a & b expand (vertices expand $\Rightarrow$ points on bars expand)

$\Rightarrow$ two copies of x locally expand

- in reality, this argument is tricky: can stay equal, to first order
- possible with strict expansiveness [see SoCG 2006 proof]

Symmetric case: adornments reflectionally symmetric about their bases



⇒ slender adornment = union of lenses

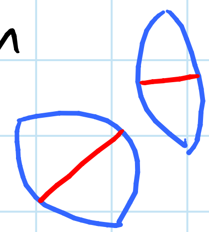
Stronger result for this case:

instantaneous expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Proof: take any two lenses of different adornments

– nonintersecting before the motion

i.e. four disks have empty intersection



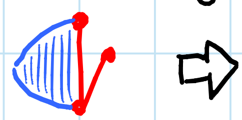
Kirszbaun's Theorem: [1934]

if we instantaneously translate n disks with an empty n-way intersection according to an expansive motion on their centers, then they still have empty intersection

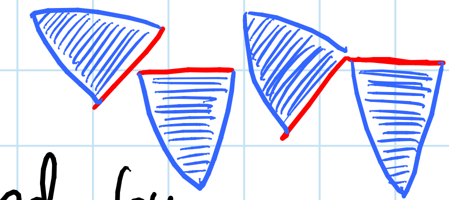
(annoying detail: Kirszbraun's disks include their boundary, but our disks might kiss — but Kirszbraun's Theorem holds for disks excluding their boundaries, by taking limits of slightly smaller disks) □

If initially intersecting, can show area of union only increases [Bezdek & Connelly 2002 ~ Kneser-Poulsen conj.]

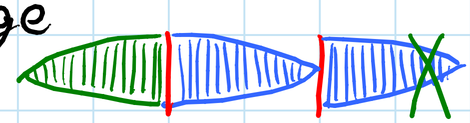
Proof that slender ⇒ not locked: (general case)

(– not true for instantaneous: )

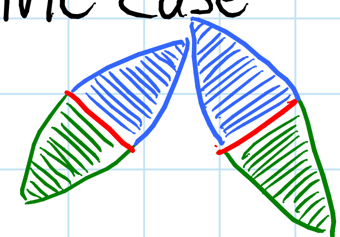
- take any 2 half-lenses of different adornments
- consider transition time from not intersecting to intersecting ⇒ touching
- 3 types of touching:



1. bases of both
   - nonintersection guaranteed by underlying chain linkage



2. base of one
   - can add symmetric lens of other, & just consider base of first (X)
   - no intersection by symmetric case

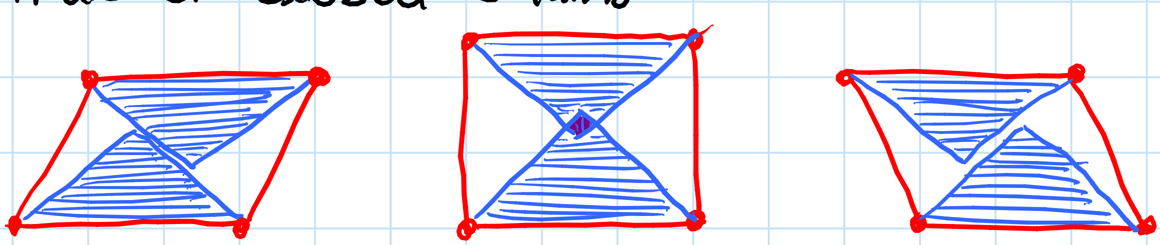


3. base of neither
   - can add symmetric lens of both
   - again no intersection by symmetric case □

Carpenter's rule theorem ⇒ straighten/convexify any slender-adorned (non-self-touching) chain

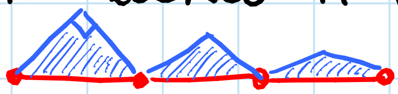
⇒ connected config. space of open chains

- not true of closed chains:

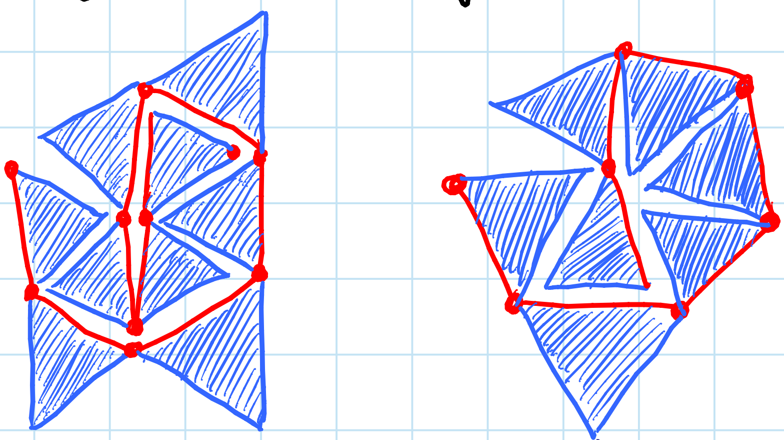


$\boxed{\text{OPEN}}$: which adornments never lock in a chain?
(like slender)

Triangles: not locked if angles opposite base $\geq 90°$
(right or obtuse)

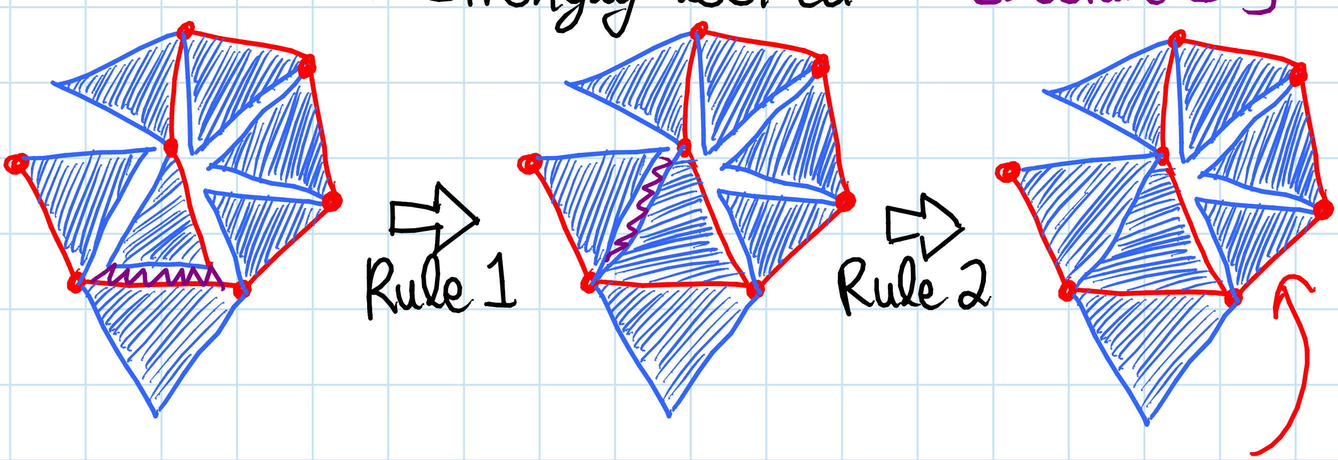


– locked (nearly) identical equilateral triangles:



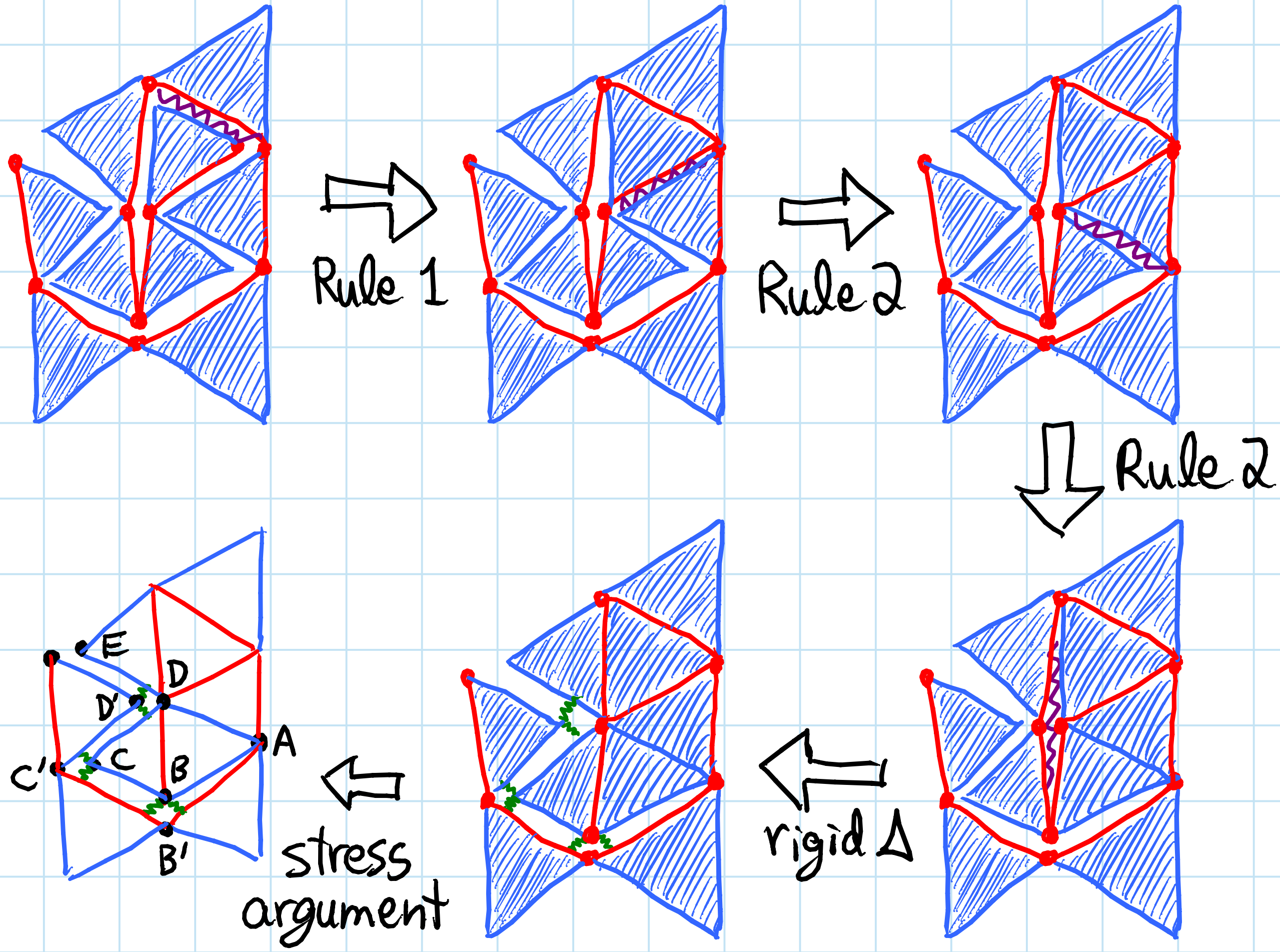
can stretch/shrink in y coord. to make locked example with any angle $< 90°$

Proof: show self-touching version rigid
$\Rightarrow$ strongly locked [Lecture 11]



Lemma: any motion of a convex polygon decreases $\geq$ two angle $\Rightarrow$ RIGID

Locked triangles proof: (cont'd)



- clearly rigid if zero-length struts were bars
- set $s(AB) = -s(AB') < 0 \Rightarrow A$ in equilibrium
- set $s(BC) = s(AB) = -s(B'C') = -s(AB') < 0$

$\Rightarrow$ force on B, B' vertical $\Rightarrow$ in equilibrium if set $s(B, AB') = s(B, B'C') < 0$ appropriately

- set $s(C'D'), s(D', DC), s(D', DE) < 0$ unique up to scale to put D' in equilibrium; scale very small
- set $s(CD) = -s(C'D') \Rightarrow D$ in equilibrium (inverse of D')
- $s(BC) < 0$ dominates $s(CD) \Rightarrow$ can set $s(C, C'B')$ & $s(C, C'D') < 0$ to put C (& hence C') in equilibrium □

$\boxed{\text{OPEN}}$: locked chain of unit squares?

Hinged dissections: [Abbott, Abel, Charlton, Demaine, Demaine, Kominers 2008/2010]

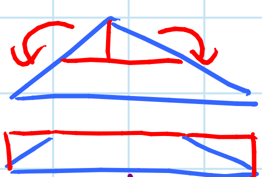
there's an open chain of hinged polygons that folds continuously into any desired finite set of polygons of equal area (without collision)

Idea:

1. start with any dissection ~ no hinges
   set of polygons that can be assembled into each target polygon
2. hinge arbitrarily (or to match one target)
3. subdivide pieces & add hinges to enable each desired assembly
4. subdivide to make pieces slender ⇒ motion

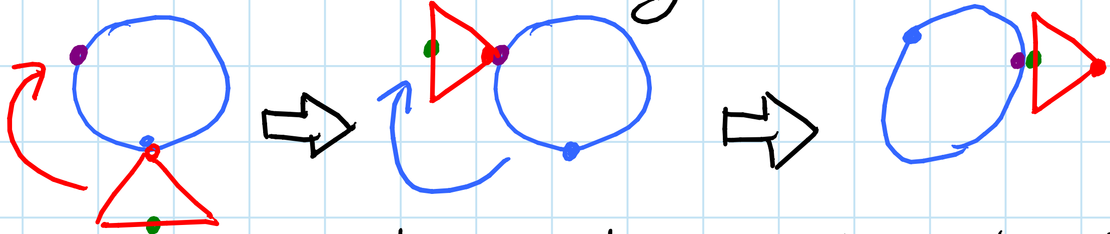
①: [Lowry 1814; Wallace 1831; Bolyai 1833; Gerwin 1833]

- cut each polygon into triangles
- cut each triangle into rectangle:



- dissect each rectangle into rectangle of height ε [Montucla 1778]
  (this is the hard step ~ skipped here)
- string rectangles from one polygon into one long height-ε rectangle
- overlay these cut patterns of $A/\varepsilon \times \varepsilon$ rect.

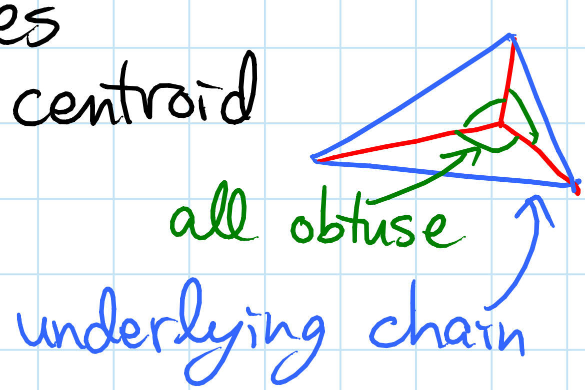
③: maintain tree hinging
- key step: effectively move rooted subtree to attach at any other vertex



- 2 of these ops. brings two vertices together
- repeat until vertices together for target
- repeat for each target polygon

pieces: without care, can roughly double for each step
- can be improved with care

④
- triangulate pieces
- cut each Δ at centroid ⇒ obtuse
- connect into chain by "walking along the outside" of the tree (Euler tour)



⇒ slender chain
⇒ not locked

6.849 Geometric Folding Algorithms: Linkages, Origami, Polyhedra
Fall 2012