First-order Theory of Concatenation

Problem 1. Let C be a 2-CM with n instructions and let Σ_n be the alphabet used in the Notes on The Semigroup Word Problem for configuration words of C. Namely,

$$\Sigma_n ::= \{\texttt{a},\texttt{b},\texttt{0},\texttt{1},\ldots,n\}.$$

(a) Explain how to write a logical formula, $F_C(x)$, for the model $(\Sigma_n^*, \cdot)$ which means that x is a con*f*iguration word for C.

(b) Explain how to write a logical formula, $S_C(x, y)$, which means that x is a configuration word for C and y is the configuration word after one *s*tep of C.

(c) Explain how to write a logical formula, $H_C(z)$, which means that z is a string representing part of a computation *h*istory of C. Namely,

$$z = z_0 z_1 \ldots z_k$$

where the z_i's are configuration words for k successive steps of C starting at the configuration represented by z_0.

(d) Explain how to write a logical formula, $L_C(x)$, which means that C ha*l*ts when started in the configuration represented by x.

(e) Conclude that there is an n such that $\overline{\text{Halts}} \leq_{\text{m}} \text{Th}((\Sigma_n^*, \cdot))$, the set of true sentences about $(\Sigma_n^*, \cdot)$. In particular, this set of sentences is not even half-decidable.

Problem 2. Show that $\text{Th}((\Sigma_n^*, \cdot)) \leq_{\text{m}} \text{Th}((\{1, 2\}^*, \cdot))$.

Problem 3. The mapping taking any nonnegative integer, $n \in \mathbb{N}$ to its binary representation is not a bijection onto the set of binary words because words with leading zeros are unused. To get a bijection, we use binary notation but with digits `1` and `2`. For example, the integer 4 would be represented as the word `12`, 5 as `21`, 6 as `22`, 7 as `111`, 8 as `112`, ... Let the empty string, λ, represent 0. In this way we do get a bijection, rep : $\mathbb{N} \to \{\texttt{1},\texttt{2}\}^*$, mapping a nonnegative integer to its unique binary representation using digits `1` and `2`.

(a) Explain how to construct a formula $C(x, y, z)$ with free variables x, y, z such that for all $i, j, k \in \mathbb{N}$,

$$C(i, j, k) \in \text{Th}((\mathbb{N}, +, \times)) \quad \text{iff} \quad \text{rep}(i) \cdot \text{rep}(j) = \text{rep}(k).$$

Hint: $C(x, y, z) \quad \text{iff} \quad x2^{\text{length}(\text{rep}(y))} + y = z.$

Solution.

$$2^{\text{length}(\text{rep}(j))} = \max \{n \mid \text{pow2}(n) \text{ and } n \leq j + 1\}.$$
$$\text{pow2}(n) \quad \text{iff} \quad \forall y.\, y \mid n \longrightarrow (y = 1 \text{ or } 2 \mid y)$$
$$n \leq m \quad \text{iff} \quad \exists y.\, n + y = m.$$

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(b) Conclude that $\text{Th}((\{\texttt{1,2}\}^*, \cdot)) \leq_{\text{m}} \text{Th}((\mathbb{N}, +, \times))$.