Problem 2

a) Traveling from x to $x+dx$ consists of both the translation in x and an accompanying translation in z by $dz = \frac{dz}{dx}dx = udx$. Therefore, the total distance is:

$$ds = \sqrt{dx^2 + dz^2} = \sqrt{1+u^2}dx.$$

The time taken to accomplish this depends on the z position, and is given by:

$$dt = \frac{ds}{z} = \frac{\sqrt{1+u^2}}{z}dx.$$

This in turn results in a total cost of:

$$J = \int_0^X \frac{\sqrt{1+u^2}}{z}dx.$$

b) The dynamics are given as:

$$\frac{dz}{dx} = u.$$

c) The Hamiltonian is given by:

$$H = g(x,u) + p(x)\frac{dz}{dx} = \frac{\sqrt{1+u^2}}{z} + p(x)u.$$

Using the fact that H must be at a minimum with respect to u allows us to write:

$$\frac{\partial H}{\partial u} = 0 = p + \frac{u}{z\sqrt{1+u^2}}$$

This gives us the value of the adjoint p:

$$p = -\frac{u}{z\sqrt{1+u^2}},$$

and thus the Hamiltonian may be written:

$$H = \frac{\sqrt{1+u^2}}{z} - \frac{u^2}{z\sqrt{1+u^2}} = \frac{1}{z\sqrt{1+u^2}}.$$

d) Knowing that H will be constant allows to obtain the differential equation:

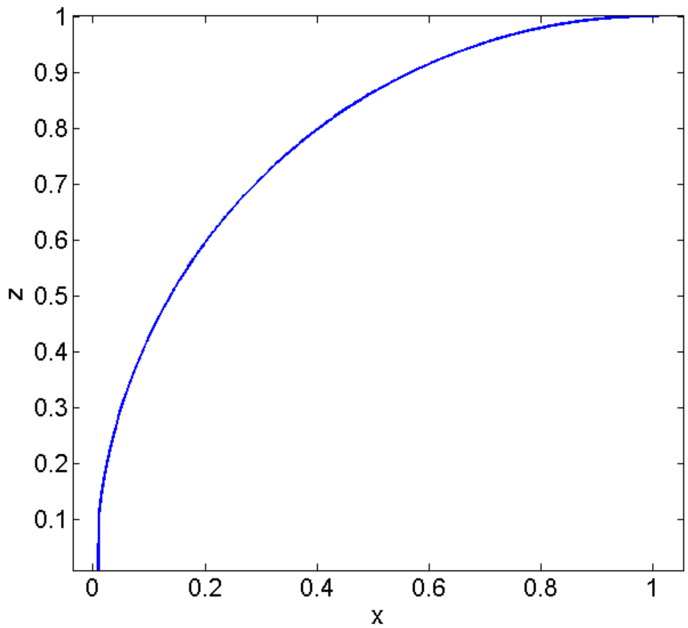
$$H = C = \frac{1}{z\sqrt{1+u^2}},$$

or

$$z^2(1+u^2) = z^2\left(1 + \frac{dz}{dx}^2\right) = D.$$

This is the differential equation defining the optimal trajectory.

e) This differential equation describes the arc of a circle. Simulating it will give a curve as shown below:



It can also be seen by looking at the equation of a circle $d_2 = (x - d_1)^2 + z^2$. Differentiating w.r.t x gives:

$$(x - d_1) + z\frac{dz}{dx} = 0, \tag{1}$$

$$(x - d_1)^2 = z^2 \frac{dz}{dx}^2, \tag{2}$$

$$d_2 - z^2 = z^2 \frac{dz}{dx}^2, \tag{3}$$

$$d_2 = z^2 \left(1 + \frac{dz}{dx}^2\right), \tag{4}$$

which is the differential equation from part d.