Modeling the Heap: Arrays and Separation Logic

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With content from the paper "Local Reasoning about Programs that Alter Data Structures" by O'Hearn, Reynolds and Yang.

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Approach 1: Heap as Array

Consider a c-style language

- New expressions: $e \coloneqq malloc(n) \mid *e$
- Statements: $c \coloneqq \ *e \coloneqq e$

In C, the heap is essentially one big array:

```
x = malloc(2);
*x = 4;
*(x+1) = z;
y = *(x) + *(x+1);
{y == 4 + z }
```

```
x = HEAP_PTR;
HEAP_PTR = HEAP_PTR + 2;
HEAP[x] = 4;
HEAP[x+1] = z;
y = HEAP[x] + HEAP[x+1];
{y == 4 + z}
```

Heap as Array

- Treat the heap as a giant array
- Use special values/ghost arrays to distinguish un-allocated memory from un-initialized memory
- Use simple counters to model the allocator
- Model using the theory of arrays

Advantages

- No new machinery required
- Very general
- Many opportunities for refinement and optimization

Heap as Array

Can even model deallocation

```
x = malloc(2);
*x = 4;
*(x+1) = z;
y = *(x) + *(x+1);
free(x);
{y == 4 + z }
```

```
x = HEAP_PTR;
LIVE[x] = true;
LIVE[x+1] = true;
SIZE[x] = 2;
HEAP_PTR = HEAP_PTR + 2;
HEAP[x] = 4;
Assert LIVE[x];
HEAP[x+1] = z;
Assert LIVE[x+1];
y = HEAP[x] + HEAP[x+1];
Assert LIVE[x] && LIVE[x+1];
Assert LIVE[x];
for(i=0; i<size[x]; ++i){
  Assert LIVE[x+i];
  LIVE[x+i] = false;
}
{y == 4 + z}
```

```
Works really well as long as
you don’t need to interact
with the user.
```

Heap as Array

What about loops?

- x points to a list of the form List{ val:int, List:next}
- At the end of the loop, t = sum(x) the sum of all elements in the list.
- How do we even express this?

```
t = 0;
while( x != null){
    t = t + *x;
    x = *(x+1);
}
```

$$\exists g.\ g[x] \wedge \forall i, j.\ \ g[i]\ \ \wedge\ \ H[i+1] = j \Leftrightarrow g[j]\ \ \wedge\ \ t = \sum_{k \in \{k | g[k]\}} H[k]$$

  - Maybe?

- What about the invariant?

Heap as Array

The approach is not entirely useless

```
t = 0;
while( x != null){
   t = t + *x;
   x = *(x+1);
}
```

- Unrolling loops can eliminates the need for invariants
- But it sacrifices soundness

```
t = 0;
if( x != null){
   t = t + *x;
   x = *(x+1);
   if( x != null){
     t = t + *x;
     x = *(x+1);
    if( x != null){
      t = t + *x;
      x = *(x+1);
      if( x != null){
       Assume false;
}  } }}
```

Heap as an Array approach

Widely used in practical tools

- One of the main drivers for scalable TOA in SMT solvers

Writing invariants can be a challenge

- Most tools that use this approach don't bother with invariants
- Problem is that the structure of any data-structure in the program gets lost in the low-level representation

Approach 2: Separation logic

- See: "Local Reasoning about Programs that Alter Data Structures"
  - By O'Hearn, Reynolds and Yang

Key idea:

- Break the heap into <u>disjoint</u> pieces
- Focus on a few small pieces at a time
- Statements affect one piece at a time

The language

- Imp + extensions
  - Stmt : $x = [e] \mid [e] = x \mid x = cons(e1, \dots, ek) \mid dispose(e)$
- Very similar in spirit to what we saw before with Op Sem
  - $s \in S : Id \to Int$
  - $h \in H : Nat \to Int$
  - $[\![C]\!] : S \times H \to S \times H \cup \{\bot\}$
  - $[\![E]\!] : S \to Int$
  - $[\![x = e]\!]\ s\ h = s\{x \to [\![e]\!]\ s\ \}\ h$
  - $[\![x = [e]]\!]\ s\ h = s\{x \to h([\![e]\!]\ s)\}\ h$
  - $[\![[e] = x]\!]\ s\ h = s\ h\{[\![e]\!]\ s \to \mathrm{s(x)}\}$
  - $[\![x = cons(e_0 \dots e_k)]\!]\ s\ h = s\{x \to j\}\ h\{j \to [\![e_0]\!]\ s, \dots, j + k \to [\![e_k]\!]\ s\}$
    $where\ j = (\max dom\ h) + 1$
  - $[\![dispose(e)]\!]\ sh = s\ h\{[\![e]\!]s \ \to \}$

Separation Logic: Notation

Heaps are described by predicates in the following language:

emp := The heap is empty

- There are no cells in this heap

$x \longmapsto y$:= The heap has exactly one cell.

- This cell is at location x
- This cell stores the value y

A * B := Heap can be partitioned into two <u>disjoint</u> regions,

- one region where A is true,
- one region where B is true

Formalizing the notation

Copied from the paper by O'Hearn, Reynolds and Yang

$$s, h \models B \quad \text{iff } [\![B]\!]s = \mathit{true}$$

$$s, h \models E \mapsto F \quad \text{iff } \{[\![E]\!]s\} = \mathit{dom}(h) \text{ and } h([\![E]\!]s) = [\![F]\!]s$$

$$s, h \models \mathtt{false} \quad \text{never}$$

$$s, h \models P \Rightarrow Q \quad \text{iff if } s, h \models P \text{ then } s, h \models Q$$

$$s, h \models \forall x.P \quad \text{iff } \forall v \in \mathtt{Ints}.\, [s \mid x \mapsto v], h \models P$$

$$s, h \models \mathtt{emp} \quad \text{iff } h = [\,] \text{ is the empty heap}$$

$$s, h \models P * Q \quad \text{iff } \exists h_0, h_1.\, h_0 \# h_1,\ h_0 * h_1 = h,\ s, h_0 \models P \text{ and } s, h_1 \models Q$$

$h_0 \# h_1 \equiv$ Domains of h_0 and h_1 are disjoint
$h_0 * h_1 \equiv$ Union of disjoint heaps

Some additional shorthand

Copied from the paper by O'Hearn, Reynolds and Yang

$$
\begin{array}{lll}
E \mapsto F_0, \ldots, F_n & \stackrel{\Delta}{=} & (E \mapsto F_0) * \cdots * (E + n \mapsto F_n) \\
E \doteq F & \stackrel{\Delta}{=} & (E = F) \wedge \mathtt{emp} \\
E \mapsto - & \stackrel{\Delta}{=} & \exists y. E \mapsto y
\end{array}
$$

An interesting property

$$(E \doteq F) * P \quad \Leftrightarrow \quad (E = F) \wedge P.$$

Algebra of heap predicates

Which assertions are valid?

$$E \Rightarrow E * E$$

$$E * F \Rightarrow E$$

$$10 \mapsto 3 \Rightarrow 10 \mapsto 3 * 10 \mapsto 3$$

$$10 \mapsto 3 \Rightarrow 10 \mapsto 3 * 42 \mapsto 5$$

$$E \mapsto 3 \Rightarrow 0 \leq E$$

$$E \mapsto - * E \mapsto -$$

$$E \mapsto - * F \mapsto - \Rightarrow E \neq F$$

$$E \mapsto - \wedge F \mapsto - \Rightarrow E = F$$

$$E \mapsto 3 * F \mapsto 3 \Rightarrow E \neq F$$

Copied from the paper by O'Hearn, Reynolds and Yang

Algebra of heap predicates

Which assertions are valid?

$$E \Rightarrow E * E$$ ✗

$$E * F \Rightarrow E$$ ✗

$$10 \mapsto 3 \Rightarrow 10 \mapsto 3 * 10 \mapsto 3$$ ✗

$$10 \mapsto 3 \Rightarrow 10 \mapsto 3 * 42 \mapsto 5$$ ✗

$$E \mapsto 3 \Rightarrow 0 \leq E$$

$$(E \mapsto -) * (E \mapsto -)$$ ✗

$$E \mapsto - * F \mapsto - \Rightarrow E \neq F$$

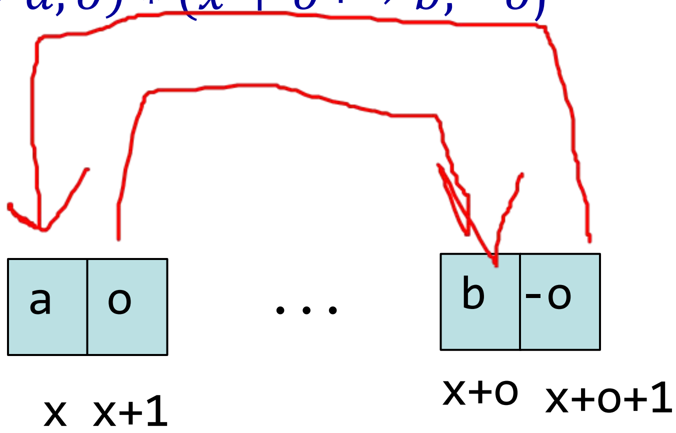
$$E \mapsto - \wedge F \mapsto - \Rightarrow E = F$$

$$E \mapsto 3 * F \mapsto 3 \Rightarrow E \neq F$$

Describing data-structures

What does this heap describe?

- $(x \longmapsto a, o) * (x + o \longmapsto b, -o)$



Proofs Rules for Separation Logic

Small Axioms

Copied from the paper by O'Hearn, Reynolds and Yang

$$\{E \mapsto -\}\, [E] := F\, \{E \mapsto F\}$$

$$\{E \mapsto -\}\, \mathtt{dispose}(E)\, \{\mathtt{emp}\}$$

$$\{x \doteq m\} x := \mathtt{cons}(E_1, ..., E_k) \{x \mapsto E_1[m/x], ..., E_k[m/x]\,\}$$

$$\{x \doteq n\}\, x := E\, \{x \doteq (E[n/x])\}$$

$$\{E \mapsto n \wedge x = m\}\, x := [E]\, \{x = n \wedge E[m/x] \mapsto n\}$$

$(x = m) \wedge emp$

Proof Rules for Separation Logic

Copied from the paper by O'Hearn, Reynolds and Yang

$$\frac{\{P\}C\{Q\}}{\{P * R\}C\{Q * R\}} \quad \text{where } \mathsf{Mod}(C) \cap \mathsf{Free}(R) = \emptyset$$

$$\frac{\{P\}C\{Q\}}{\{P \wedge R\}C\{Q \wedge R\}} \quad \text{where } \mathsf{Mod}(C) \cap \mathsf{Free}(R) = \emptyset$$

note: $\mathsf{Mod}(x = e) = \{x\}, \mathsf{Mod}([e] = x) = \emptyset$

$$\frac{\{P\}C\{Q\}}{\{\exists x. P\}C\{\exists x. Q\}} \quad x \notin Free(C)$$

$\mathrm{Free}(P)$ is the set of variables occurring freely in P

More Cycle Free Data-structures

Copied from the paper by O'Hearn, Reynolds and Yang

Linked lists.

$$\begin{aligned}
\mathsf{lseg}(e, f) &\Leftrightarrow \text{if } e = f \text{ then emp else} \\
&\quad \exists y. e \mapsto -, y * \mathsf{lseg}(y, f) \\
\mathsf{list}(e) &\Leftrightarrow \mathsf{lseg}(e, \mathsf{nil})
\end{aligned}$$

$$\mathsf{lseg}(x, y) * \mathsf{lseg}(y, x)$$

$$\mathsf{lseg}(x, t) * t \mapsto -, y * \mathsf{list}(y)$$

Examples

Proof of $\{\mathsf{list}(x)\}y = \mathsf{cons}(b, x)\{\mathsf{list}(y)\}$

$$\frac{}{\{\mathsf{list}(x)\}y = \mathsf{cons}(b, x)\{\mathsf{list}(y)\}}$$

Examples

Proof of $\{\mathsf{list}(x) \wedge x \neq \mathsf{nil}\} t = [x] \{x \mapsto t * \mathsf{list}(t)\}$

$$\frac{}{\{\mathsf{list}(x) \wedge x \neq \mathsf{nil}\} t = [x] \{x \mapsto t * \mathsf{list}(t)\}}$$

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