Fluids - B

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Outline

> **Review of last time**

> **Poiseuille flow**

> **Stokes drag on a sphere**

> **Squeezed-film damping**

> **Electrolytes & Electrokinetic separations**

Last time

> **Surface Tension**
>   - **Force at a liquid-fluid interface**

$$\tau = \eta \frac{\partial U_x}{\partial y}$$

> **Viscosity**
>   - **Constitutive property relation shear stress to shear rate**

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$$

> **Material Derivative**
>   - **Time derivative taking into effect convection**

$$\frac{D\rho_m}{Dt} + \rho_m \nabla \cdot \mathbf{U} = 0$$

> **Mass continuity**

> **Navier-Stokes Equation**
>   - **Fundamental relation for Newtonian fluids**

$$\rho_m \frac{D\mathbf{U}}{Dt} = -\nabla P + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla(\nabla \cdot \mathbf{U}) + \rho_m \mathbf{g}$$

> **Reynolds Number**
>   - **The MOST IMPORTANT dimensionless number**

$$Re = \frac{\rho_m L U}{\eta}$$

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Poiseuille Flow

> **Pressure-driven flow through a pipe**

  - **In our case, two parallel plates**

> **Velocity profile is parabolic**

> **This is the most common flow in microfluidics**

  - **Assumes that h<<W**
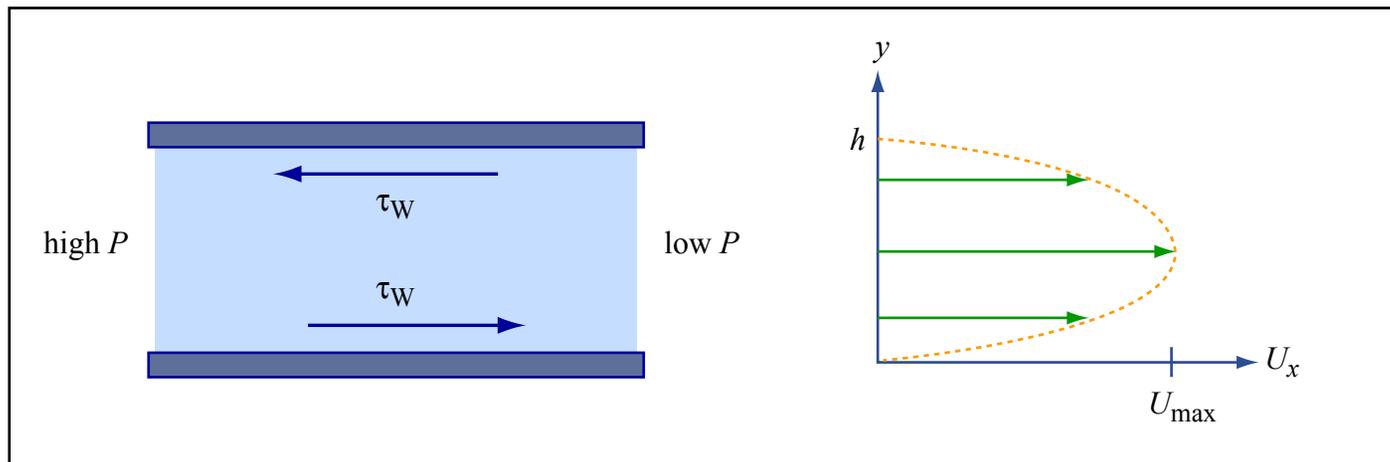
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Adapted from Figure 13.5 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 329. ISBN: 9780792372462.

Solution for Poiseuille Flow

> **Assume**

- **Incompressible**
- **Steady**
- **U_x only depends on y**
- **Ignore gravity**

$$\rho_{\mathrm{m}}\left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U}\right) = -\nabla P + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3}\nabla\left(\nabla \cdot \mathbf{U}\right) + \rho_m \mathbf{g}$$

> **Assume a uniform pressure gradient along the pipe**

> **Result is Poisson's eqn**

> **Boundary conditions:**

- **Relative velocity goes to zero at the walls**
  - » **no-slip boundary condition**

$$\frac{dP}{dx} = -K$$

$$\frac{\partial^2 U_x}{\partial y^2} = -\frac{K}{\eta}$$

Solution for Poiseuille Flow

> **Integrate twice to get solution**

> **Maximum velocity is at center**

> **Can get linear flowrate [m/s] and volumetric flowrate [m³/s]**

> **Can get lumped resistor using the fluidic convention**

> **Note STRONG dependence on *h***

> **This relation is more complicated when the aspect ratio is not very high…**

$$U_x = \frac{1}{2\eta}\Big[y(h-y) \Big] K$$

$$U_{max} = \frac{h^2}{8\eta} K$$

$$Q = W \int_0^h U_x \, dy = \frac{Wh^3}{12\eta} K$$
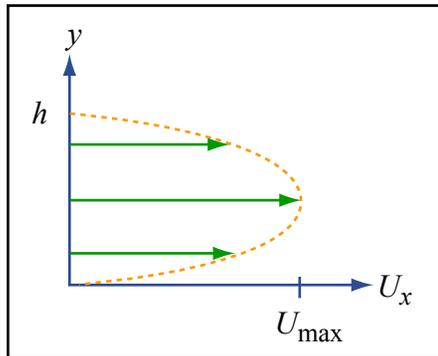
$$\Delta P = \text{effort} = KL$$

$$\Delta P = \frac{12\eta L}{Wh^3} Q$$

$$\Rightarrow R_{Pois} = \frac{12\eta L}{Wh^3}$$

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Adapted from Figure 13.5 in Senturia, Stephen D. *Microsystem Design*.
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Development Length

> **It takes a certain characteristic length, called the development length, to establish the Poiseuille velocity profile**

> **This development length corresponds to a development time for viscous stresses to diffuse from wall**

> **Development length is proportional to the characteristic length scale and to the Reynolds number, both of which tend to be small in microfluidic devices**

$$time \approx L^2 \Big/ \eta^* \approx \mathrm{Re}\frac{L}{U}$$

$$L_D \approx (time)U \approx \mathrm{Re}\, L$$

A note on vorticity

> **A common statement is to say that laminar flow has no vorticity**

> **What is meant is that laminar flow has no turbulence**

> **Vorticity and turbulence are different**

> **Can the pinwheel spin?**
  - **Then there is vorticity**

> **Demonstrate for Poiseuille flow**

$$\boldsymbol{\omega} = \nabla \times \mathbf{U}$$
Vorticity

$$U_x = \frac{1}{2\eta}\left[y(h-y)\right]K$$

$$\boldsymbol{\omega} = \mathbf{n_y}\frac{\partial U_x}{\partial z} - \mathbf{n}_z\frac{\partial U_x}{\partial y}$$

$$\boldsymbol{\omega} = -\mathbf{n}_z\frac{K}{2\eta}(h-2y)$$

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Stokes Flow

> **Steady-state flow in which inertial effects can be neglected, Re→0**

> **The result is a vector Poisson equation**

> **Also called "creeping flow"**

> **Action is instantaneous**
> - **No "mass" in system**
> - **Incompressible: no springs**

> **This is a typical approximation made in microfluidics**

When $\rho_m \dfrac{D\mathbf{U}}{Dt}$ can be neglected

$$\eta \nabla^2 \mathbf{U} = \nabla P *$$

Stokes' drag on a sphere

> **In creeping flow, one can solve for the flow field around a sphere placed in an initially uniform flow field**

> **This can be used to find the stresses on the sphere and sum them to find the total drag**

> **This is called the Stokes' drag**

> **This is often the predominant particle force in microfluidic systems**

> **See Deen's text for derivation**
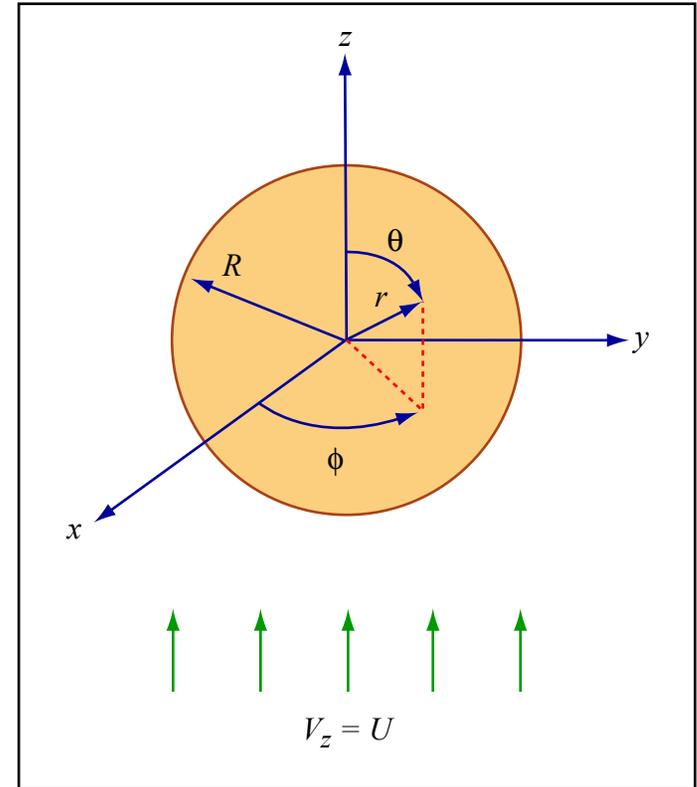
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$$F_d = 6\pi\eta R U$$

Stokes' drag on a sphere

> **This is strictly valid ONLY in a uniform flow**

> **These are hard to make…**

> **Instead, we can use $U_x(y)$ of the parabolic flow profile to calculate a height-dependent drag force**

> **This approach fails when the particle is too big**

> **Instead, take advantage of published solutions**
>   - **Shear flow: Goldman et al., *Chem. Eng. Sci.* 22**, 653 (1967).
>   - **Poiseuille flow: Ganatos et al. *J. Fluid Mech.* 99**, 755 (1980).
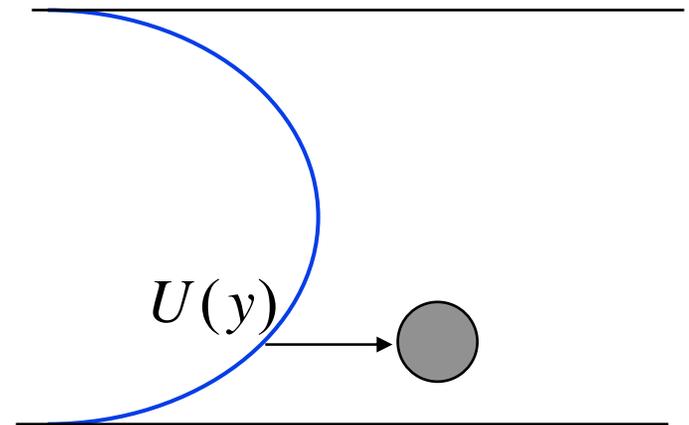
$$U(y)$$

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Squeezed-Film Damping

> This is how we will get our *b* (or *R*) for the parallel-plate actuator

> The result of motion against a fluid boundary
  - If the fluid is incompressible, there can be a large pressure rise, so large back forces result
  - If the fluid is compressible, it takes finite motion to create a pressure rise

> In either case, the dissipation due to viscous flow provides a damping mechanism for the motion

> This is related to "lubrication theory"

F

Moveable

$h(t)$

Fixed
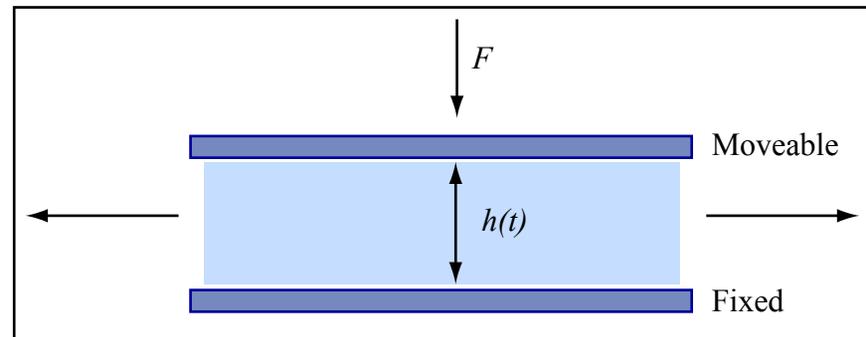
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Adapted from Figure 13.6 in Senturia, Stephen D. *Microsystem Design*. Boston,
MA: Kluwer Academic Publishers, 2001, p. 333. ISBN: 9780792372462.

The Reynolds Equation

> **Assumptions for compressible isothermal squeezed-film damping**

- **One-dimensional pressure gradient:** $P(r,t) = P(y,t)$ **only**
  - » **No pressure gradient in** z **or along plate** (x)
- **Stokes flow**
- **Poiseuille flow profile in the plane**
- **Ideal gas law**
- **Isothermal (temperature rise due to compression is small, and heat flow to the walls is rapid)**
- **No-slip BC's**
- **Rigid plate:** $h(r,t) = h(t)$ **only**
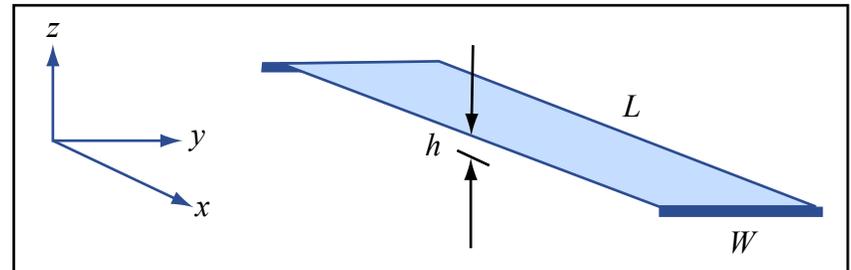
> **The result is a version of the** **Reynolds equation**

$$\frac{\partial(Ph)}{\partial t} = \frac{h^3}{12\eta}\left(\frac{1}{2}\nabla^2 P^2\right)$$

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Adapted from Figure 13.7 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 334. ISBN: 9780792372462.

Example: Rigid Plate Damping

> **Now, assume small motions**
>   **➔ Linearize**

> **The result is (guess what!) the heat-flow equation**

$$\frac{\partial(Ph)}{\partial t} = \frac{h^3}{12\eta}\left(\frac{1}{2}\nabla^2 P^2\right)$$

If we linearize:

$$h = h_0 + \delta h \quad P = P_o + \delta P$$
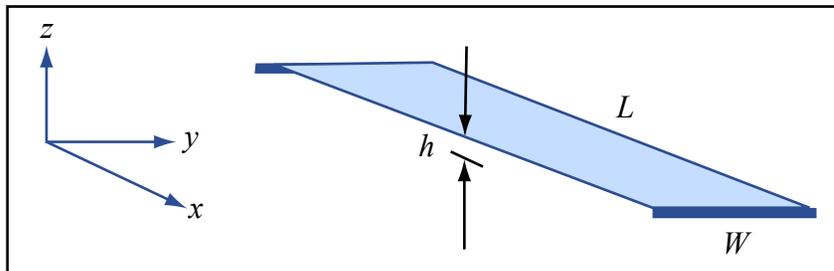
Normalize: $\xi = \dfrac{y}{W} \quad \hat{p} = \dfrac{\delta P}{P_o}$

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Adapted from Figure 13.7 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 334. ISBN: 9780792372462.

$$\frac{\partial \hat{p}}{\partial t} = \frac{h_0^2 P_o}{12\eta W^2}\frac{\partial^2 \hat{p}}{\partial \xi^2} - \frac{\dot{h}}{h_o}$$

Suddenly Applied Motion

> **We already solved a very similar problem**
> - **Impulse of heat into 1-D resistor**

> **Now we have a velocity impulse: sudden change of height**

> **We get a series of 1st-order terms, as before**

> **Only need 1st term, which is an RC circuit**

> **R for viscous damping**

> **C for gas compressibility**

> **Details are in the book**

$$F(s) = \left[\frac{96\eta LW^3}{\pi^4 h_o^3} \sum_{n\ odd} \frac{1}{n^4} \frac{1}{1 + \dfrac{s}{\alpha_n}} \right] sz(s)$$

$$\alpha_n = \frac{n^2\pi^2 h_o^2 P_o}{12\eta W^2} = \omega_n$$

$n=1$

$$b = \frac{96\eta LW^3}{\pi^4 h_o^3}$$

$$\omega_c = \frac{\pi^2 h_o^2 P_o}{12\eta W^2}$$

$R = b$ $C = \dfrac{1}{b\omega_c}$
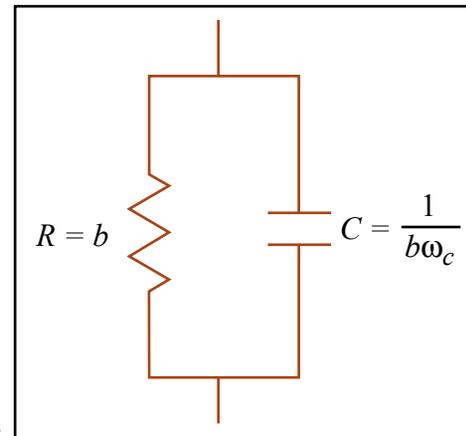
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> **Squeezed-film damping**

> **Electrolytes & Electrokinetic separations**

Electrokinetic Phenomena

> **It is a coupled-domain problem in which electrostatic forces result in fluid flow (and vice versa)**

> **Start with electrolytes, move into double layer, and finally show how to manipulate the double layer**

> **It is the driver behind ALOT of early micro-TAS work**

Electrolytes

> **Electrolytes: liquids with mobile ions**

  - **Examples: water, PBS**

> **Ions can move via concentration gradients (diffusion) or electric fields (drift)**

> **Macroscopically, the liquid is approximately charge-neutral (called quasineutrality)**

Ion mobility (cm²/V-s)

$$N_i = \frac{z_i}{|z_i|} u_i C_i E - D_i \nabla C_i$$

Flux (cm⁻²-s⁻¹)

Valence

Concentration (cm⁻³)

Diffusivity (cm²/s)

$$\rho_e = \sum_i z_i q_e C_i \approx 0 \text{ in the bulk}$$

In neutral regions: $\nabla^2 \phi = \dfrac{-\rho_e}{\varepsilon} = 0$

For a binary electrolyte (e.g., NaCl in water) $\longrightarrow$

$$N_+ = u_+ C_+ E - D_+ \nabla C_+$$
$$N_- = -u_- C_- E - D_- \nabla C_-$$
$$\rho_e = q_e \left(C_+ - C_- \right) \approx 0$$

Electrolytes

> **Surfaces with fixed charge can lead to net space charge in the liquid**

> **Diffusion competes with drift, and at equilibrium, Boltzmann distribution follows**

> **This leads to the Poisson-Boltzmann equation**
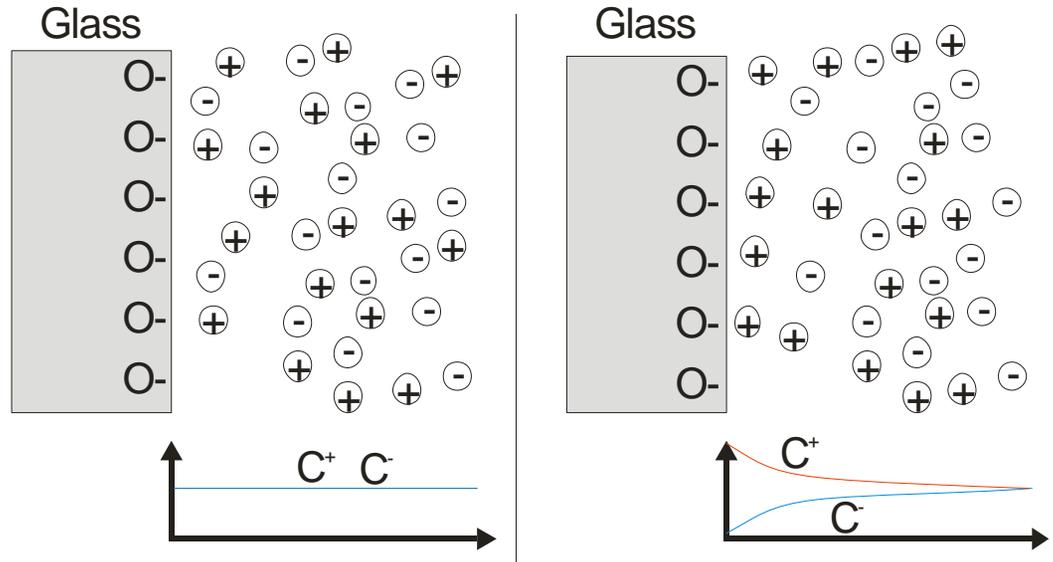
$$N_i = \frac{z_i}{|z_i|} u_i C_i E - D_i \nabla C_i = 0 \quad \text{at equilibrium}$$

$$\Rightarrow C_i(x) = C_{i,o} e^{-\frac{z_i q_e (\phi(x) - \phi_o)}{k_B T}}$$

Near the wall: $\quad \nabla^2 \phi = -\frac{\rho_e}{\varepsilon} \quad \Rightarrow \quad \boxed{\nabla^2 \phi = -\frac{1}{\varepsilon} \sum_i z_i q_e C_{i,o} e^{-\frac{z_i q_e (\phi - \phi_o)}{k_B T}}}$

Diffuse Double Layer

Express in terms of $\hat{\phi} = \phi - \phi_0$:
$$\nabla^2 \hat{\phi} = -\frac{1}{\varepsilon} \sum_i z_i q_e C_{i,o} e^{-\frac{z_i q_e \hat{\phi}}{k_B T}}$$

Expand for small potential variations

$$\nabla^2 \hat{\phi} \cong -\frac{1}{\varepsilon} \sum_i z_i q_e C_{i,o} + \frac{q_e^2}{\varepsilon k_B T}\left[\sum_i z_i^2 C_{i,o}\right]\hat{\phi}$$

Assuming the reference region is charge neutral

$$\nabla^2 \hat{\phi} = \frac{1}{L_D^2}\hat{\phi}$$

$$\hat{\phi} = \phi_w e^{\pm z/L_D}$$

For binary monovalent electrolyte (e.g., NaCl)
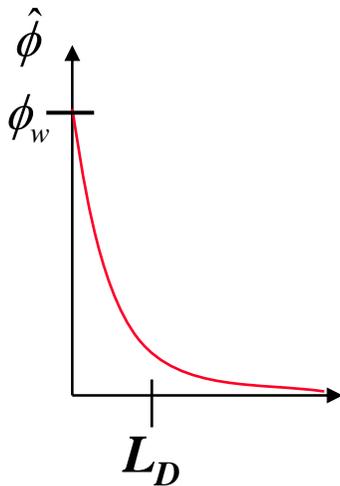
Debye Length

$$\frac{1}{L_D} = \sqrt{\frac{q_e^2}{\varepsilon k_B T}\left[\sum_i z_i^2 C_{i,o}\right]} = \sqrt{\frac{2 q_e^2 C_{i,o}}{\varepsilon k_B T}}$$

$$L_D \sim \frac{1}{\sqrt{C_{i,o}}}$$

L_D=1 nm for 0.1 M NaCl

Double-layer charge

> **We can calculate the total charge in the double layer**

> **It must balance the charge at the wall**
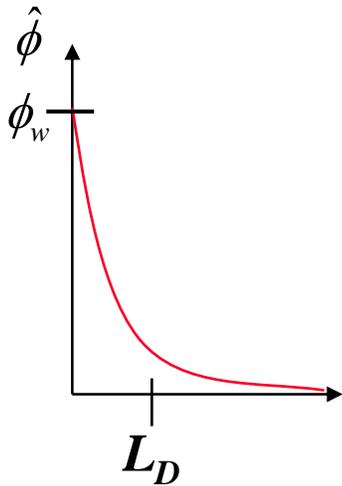
Near the wall: $\hat{\phi} = \phi_w e^{-\frac{z}{L_D}}$

The charge density is

$$\nabla^2 \hat{\phi} = -\frac{\rho_e}{\varepsilon} \Rightarrow \rho_e = -\frac{\varepsilon}{L_D^2} \hat{\phi}$$

Total charge per unit area
in diffuse layer

$$\sigma_d = \int_0^\infty \rho_e \, dz = -\frac{\varepsilon \phi_w}{L_D} = -\sigma_w \quad \Rightarrow \quad \hat{\phi} = \frac{\sigma_w L_D}{\varepsilon} e^{-\frac{z}{L_D}}$$

DL charge/area [C/cm²]

wall charge/area [C/cm²]

Actual Double Layers

> **The actual situation is MUCH more complicated**

> **Some ions are tightly or specifically adsorbed, forming the Stern layer and screening the wall charge**

> **The rest distribute in a diffuse double-layer: the Gouy-Chapman layer**

> **This is an active area of research**
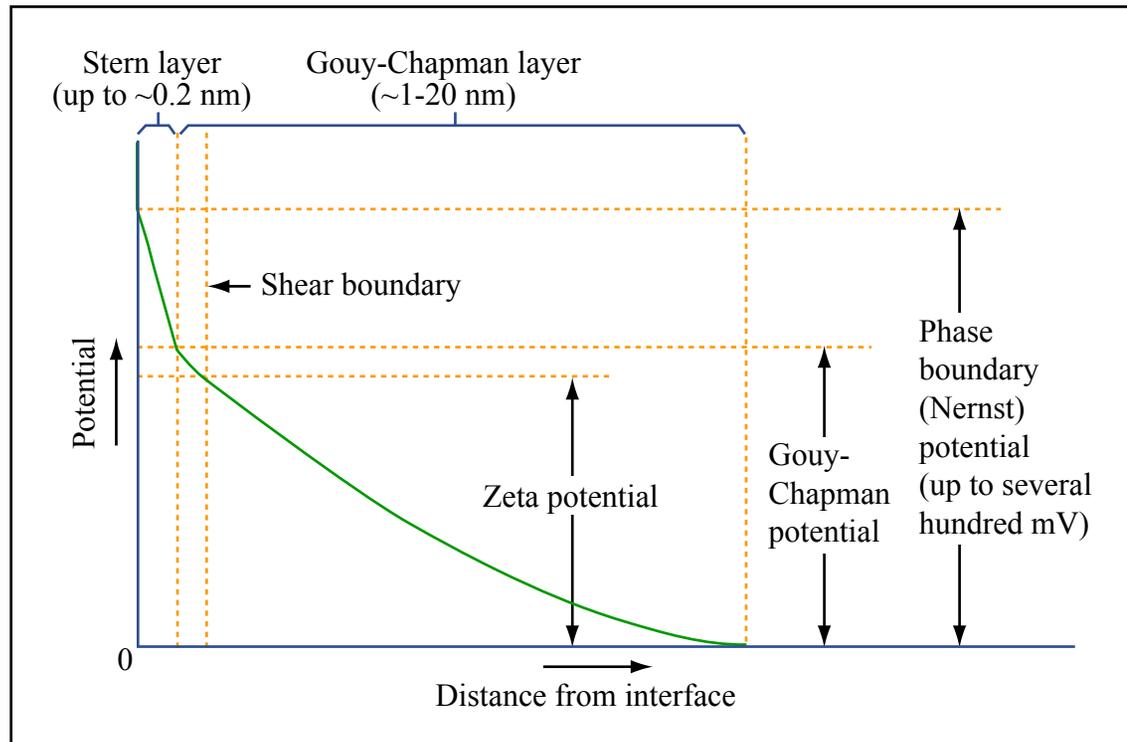
Image by MIT OpenCourseWare.

Electroosmotic Flow

> **An axial electric field exerts a force on the charge in the diffuse double layer, which drags the fluid down the pipe**

Insulating solid

Electrolyte

Electric field

Net charge in diffuse layer

Insulating solid

Image by MIT OpenCourseWare.

Analysis of Electroosmotic Flow

> **Assume**

- **Creeping flow**
- **One dimensional flow** $U_x(z)$
- **No pressure drop**
- **Electrical body force**

> **Express charge density in terms of wall surface charge density**

> **Set up differential equation**

$$\eta \nabla^2 U_x = \rho_e E_x$$
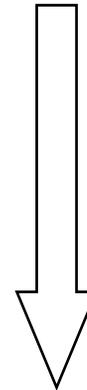
$$\rho_e = -\frac{\varepsilon}{L_D^2} \hat{\phi}$$

$$= -\frac{\sigma_w}{L_D} e^{-\frac{z}{L_D}}$$

$$\frac{d^2 U_x}{dz^2} = \left(\frac{\sigma_w}{\eta L_D} e^{-\frac{z}{L_D}} \right) E_x$$

Analysis of Electroosmotic Flow

> **This leads to a PLUG-FLOW profile**

$$\frac{d_2 U_x}{dz^2} = \left(\frac{\sigma_w}{\eta L_D} e^{-\frac{z}{L_D}} \right) E_x$$

Integrate twice, and use boundary conditions:

$$U_x \left(z = \delta \right) = 0 \qquad \hat{\phi}(\delta) = \zeta \qquad \left. \frac{dU_x}{dz} \right|_{h/2} = 0$$

$$U_x = -\frac{\sigma_w L_D}{\eta} \left(e^{-\frac{\delta}{L_D}} - e^{-\frac{z}{L_D}} \right) E_x$$

$$\phi \left(z \right) = \frac{\sigma_w L_D}{\varepsilon} e^{-\frac{z}{L_D}}$$

$$U_x = -\frac{\varepsilon}{\eta} \left(\zeta - \hat{\phi} \right) E_x$$
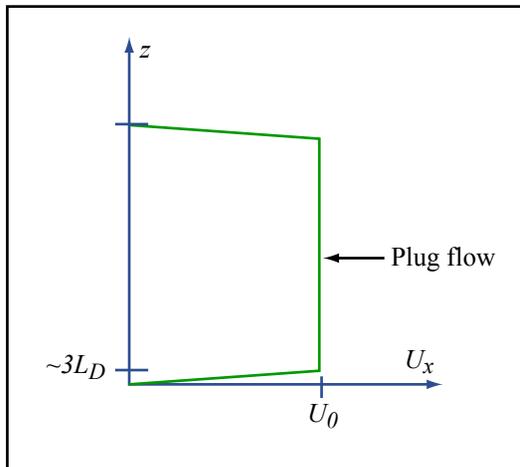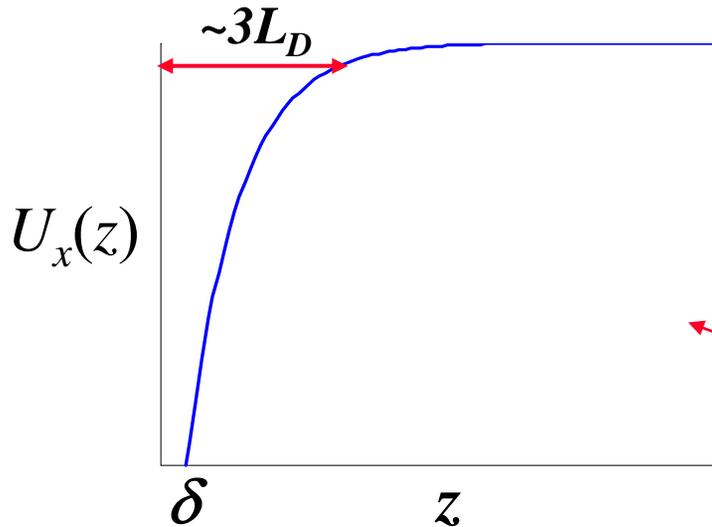
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Analysis of Electroosmotic Flow

> **This leads to a PLUG-FLOW profile**

> **Flow depends on zeta potential ζ**

- **The potential at the slip plane δ**
- **Which is in a different place than the wall, the Stern layer, or L_D**
- **It is what is measured experimentally**
- **For $h >> L_D$, one typically assumes**

$$U_x = -\frac{\varepsilon}{\eta}\left(\zeta - \hat{\phi}\right)E_x$$

$$U_0 = -\frac{\sigma_{\mathrm{w}} L_D}{\eta} e^{-\frac{\delta}{L_D}} E_x = -\frac{\varepsilon \zeta}{\eta} E_x$$

$$\text{for } z > 3L_D$$

$$\delta = 0$$

$$\phi_w = \zeta$$

Electrophoresis

> **This is just electroosmosis around a solid surface**

> **Each ionic species has its own mobility**

> **Therefore, in an electrolyte in which there is a net electric field, ions will drift at various rates**

> **This is the basis of a separation technology called electrophoretic separation**

$$U_{ep} = \left(\frac{\varepsilon \zeta}{\eta} \right) \mathsf{E}_x = \mu_{ep} \mathsf{E}_x$$

For a "large" particle

More generally

Electrokinetic separation

> **This was THE original driver for micro-TAS**

- • **TAS = total analysis systems**

> **Create fully integrated microsystems that would go from "sample to answer"**

> **The KEY enabler was the integration of non-mechanical valves with the separation column**

- • **This creates extremely narrow sample plugs**

> **Also important is the ability to multiplex**

Image removed due to copyright restrictions.

> **The *actual* sample prep was (and is) usually ignored**

> **This is/was the raison d'être of Caliper & Aclara**

> **Aclara died, unclear if Caliper will succeed**

Electrokinetic separation

> We use a channel-crossing structure to select a sample plug

> Then we switch voltages to drag the plug down a separation column

> In one approach, EO and EP co-exist, but EO dominates

> Different species travel at different rates ➔ separation
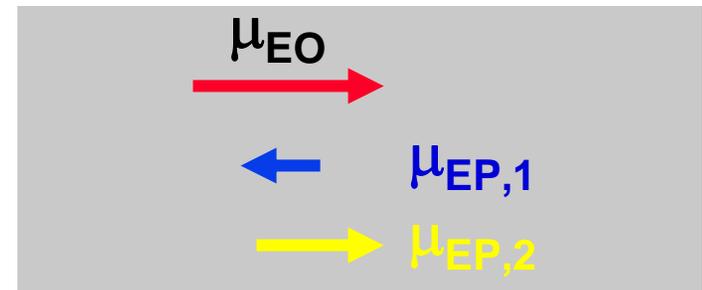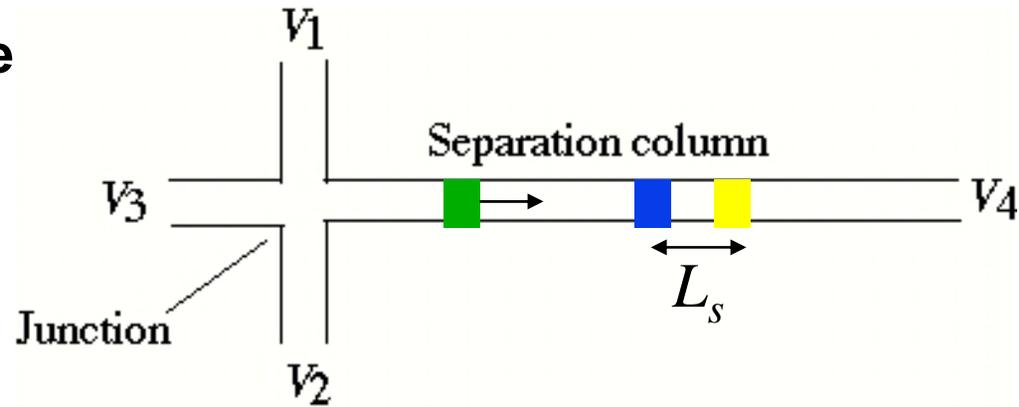
$$L_s = t_{sep} \left| \mu_{EP,1} - \mu_{EP,2} \right| \mathsf{E}_x$$

Separation time

Schematic Illustration

> **The key discovery was that liquid samples could be controlled with voltages**

> **This allows one to valve and pump liquids**
> - **Create small sample plugs**

ORNL movie
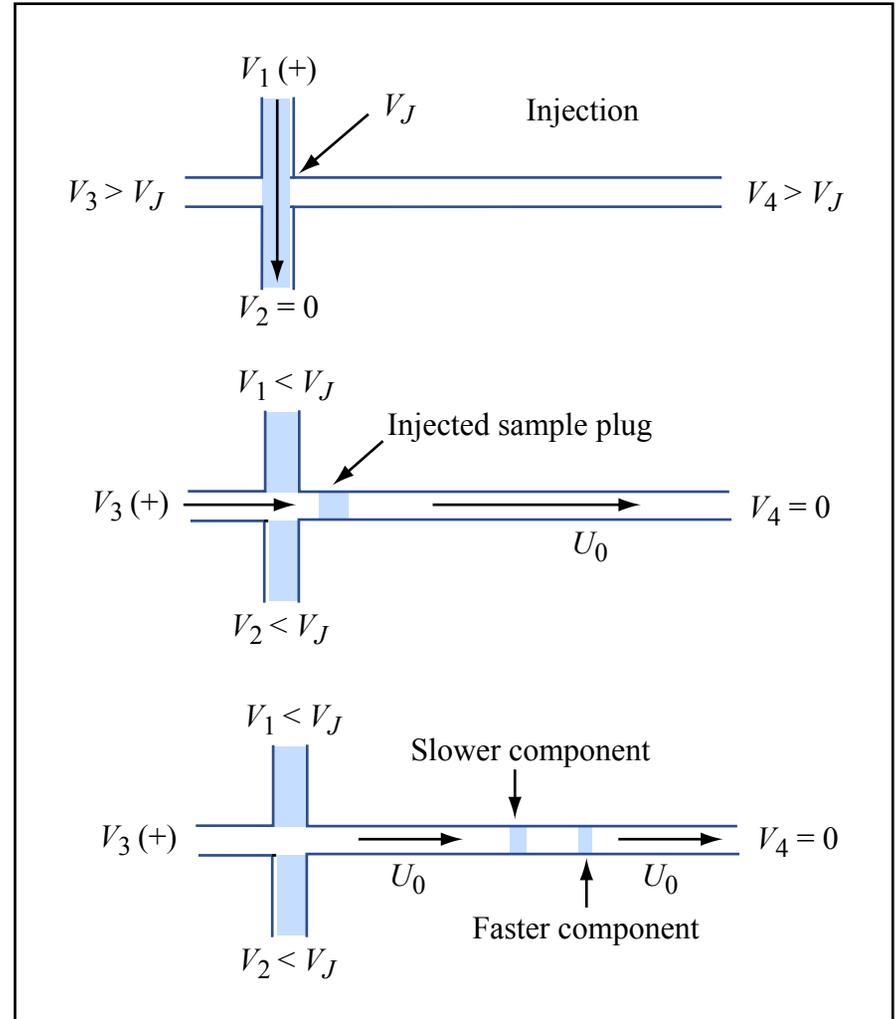
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Electrokinetic separation

> **The "macro" technology is "conventional" capillary electrophoresis**

  - **Most notably used in the human genome project**

> **Sample loading is the problem**

> **To separate two species (ignoring diffusion), we need**

$$L_s \geq W$$

> **Smaller starting *W* means we can use a shorter channel**

> **And get a faster separation**
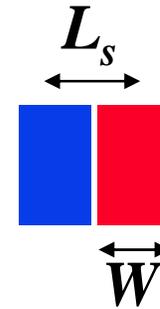
Electrokinetic separation

> **Conventional capillaries have larger W because injection is not integrated**

- **Though this is always getting better**

> **This thus requires a longer channel**

> **Microfab also allows for integration**

Richard Mathies (UCB)
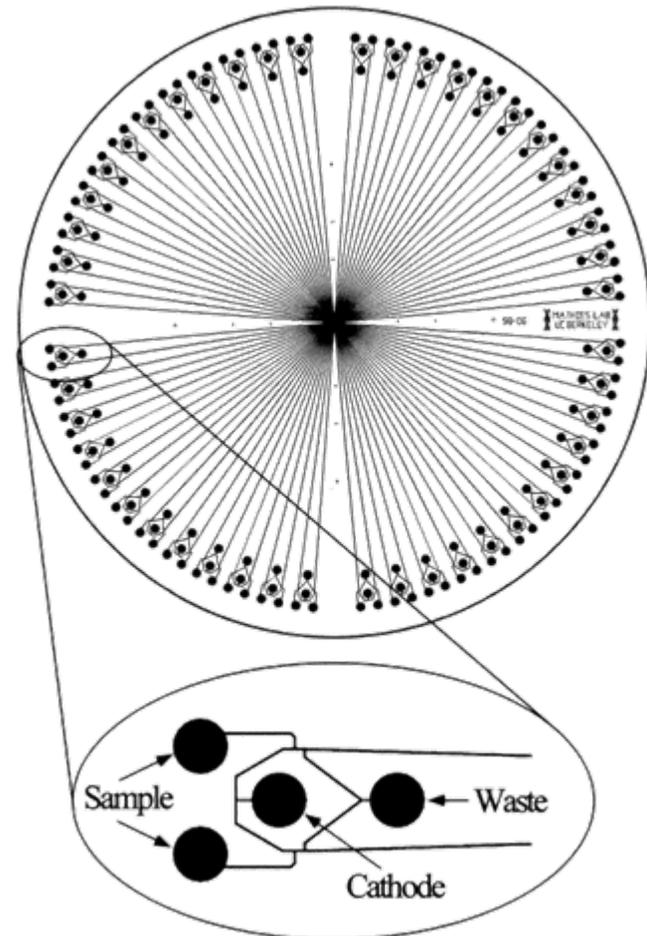
Courtesy of Richard A. Mathies. Used with permission.

An aside on electrodes

> **Current in the electrolyte is carried by ions**

> **Current in the wire is carried by electrons**

> **At the surface, something must happen to transfer this current**

> **This is electrochemistry and the typical byproduct are gases → bubbles**

Electrokinetics

> **That is just the beginning**

> **Dielectrophoresis**

- **Force on dipoles in non-uniform electric fields**
- **Can use AC fields**
- **Can hold things in place**

> **Other phenomena**

- **Electrohydrodynamics**
- **Electrowetting**
- **Induced-charge electrophoresis/electroosmosis**

Comparing EOF & pressure-driven flow

> **If you are designing a chip that needs to move liquids around, which method is best?**

> **Issues to consider**

- **Flowrate scaling**
- **Liquid composition**
- **System partitioning**
- **Materials**
- **Species Transport**

Comparing EOF & pressure-driven flow

> **Flowrate**

- **Water in rectangular SiO_2 channel**
- **h varies, W=1000 μm, L=2 cm, ε=80ε_0, ζ=50 mV**
- **Drive with E_x=100 V/cm, ΔP=5 psi**

$$U_0 = -\frac{\varepsilon\zeta}{\eta}E_x$$

$$Q = \frac{Wh^3}{12\eta L}\Delta P$$

$$Q = U_0 hW = -\frac{\varepsilon\zeta hW}{\eta}E_x$$

$$U_0 = \frac{Q}{A} = \frac{h^2}{12\eta L}\Delta P$$

Comparing EOF & pressure-driven flow

> ## Scaling

- **Pressure-driven flow larger in large channels**
  - » **Due to cubic dependence of flow resistance**
- **EO flow larger in small channels**

> ## Both scale equivalently with channel length

- **Larger _L_ requires more voltage and higher pressure to get same flow**
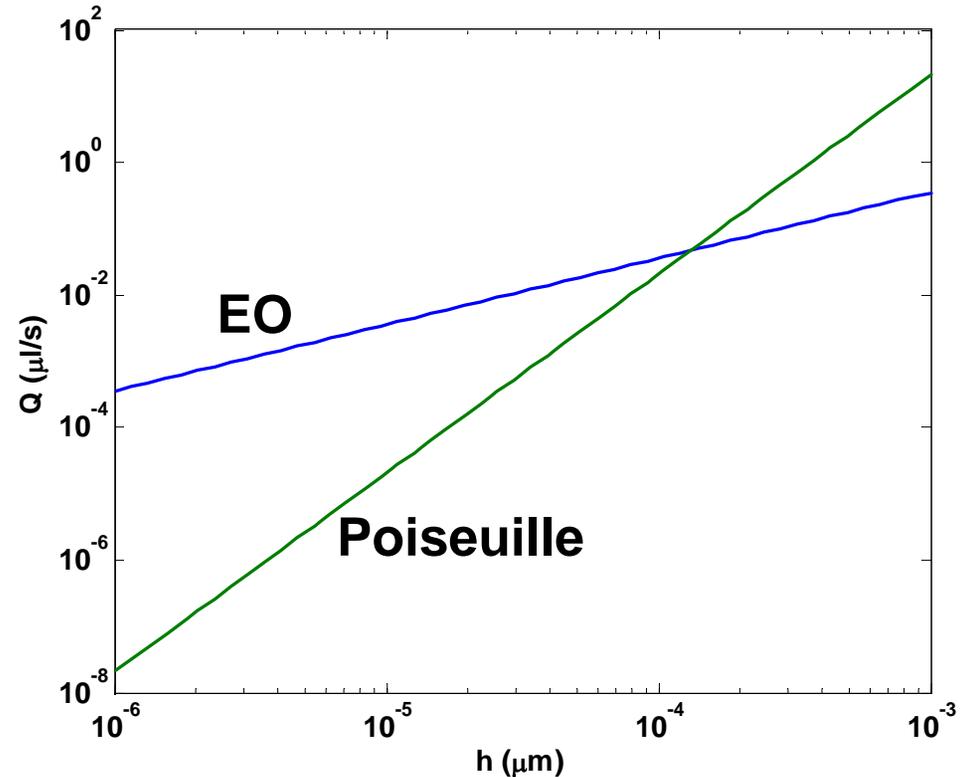
Comparing EOF & pressure-driven flow

> **Other issues typically matter more**

> **Valving**
- **EOF has "built-in" valving using E-fields**
- **Poiseuille needs mechanical valves**

> **Liquid limitations**
- **Poiseuille flow can pump any liquid**
- **EOF ionic strength limits**
  - » **Debye length depends on $1/\sqrt{C_0}$**
  - » **Increasing ionic strength decreases L_D and thus EOF**
  - » **Typically use ~10-100 mM salt buffer**
- **EOF pH limits**
  - » **pH affects wall charge → affects EOF**
  - » **Typically use pH ~7 buffers**

Comparing EOF & pressure-driven flow

> **Materials issues**

- **Poiseuille flow can use any material**
- **EOF requires defined (and stable) surface charge**
    - » **Best is silica (or at least glass)**
    - » **Polymers are more difficult**
    - » **Surface charge can change as molecules adsorb, etc.**

> **System partitioning**

- **Both approaches involve off-chip components**
    - » **Electrodes for EOF**
    - » **Pumps for Poiseuille flow**
- **Integrating complete lab-on-a-chip usually BAD idea**
    - » **EOF: Easier to use external electronics**
    - » **Poiseuille: Pumps hard to make on-chip**

Comparing EOF & pressure-driven flow

> **Transport limitations**

- **EOF**

  » **Will separate molecules as they are convected downstream**

  » **Species in flow must tolerate E-fields (DNA/proteins OK, cells not so good)**

  » **Plugs remain plugs**

- **Poiseuille flow**

  » **Will not separate molecules**

  » **Objects in flow must tolerate shear/pressure (DNA/proteins OK, cells OK depending on shear/pressure)**

  » **Will distort plugs**

What's next

> **We will discuss the behavior of the stuff in the liquids**

> **How to manipulate that stuff at the microscale**

> **And then we head into system-level issues**
- **Feedback, Noise, etc.**