Fluids A

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***(with thanks to SDS)**

Outline

- Intro to microfluidics
- Basic concepts: viscosity and surface tension
- Governing equations
- Incompressible laminar flow

Microfluidics

> **The manipulation and use of fluids at the microscale**

> **Most fluid domains are in use at the microscale**

- **Explosive thermofluidic flows**
  - » **Inkjet printheads**
  - » **We will not cover this regime**
- **High-speed gas flows**
  - » **Micro-turbomachinery**
  - » **We will not cover this regime**
- **Low-speed gas flows**
  - » **Squeeze-film damping**
  - » **We'll do a bit of this to get *b* for SMD**
- **Liquid-based slow flow**
  - » **This will be the focus**

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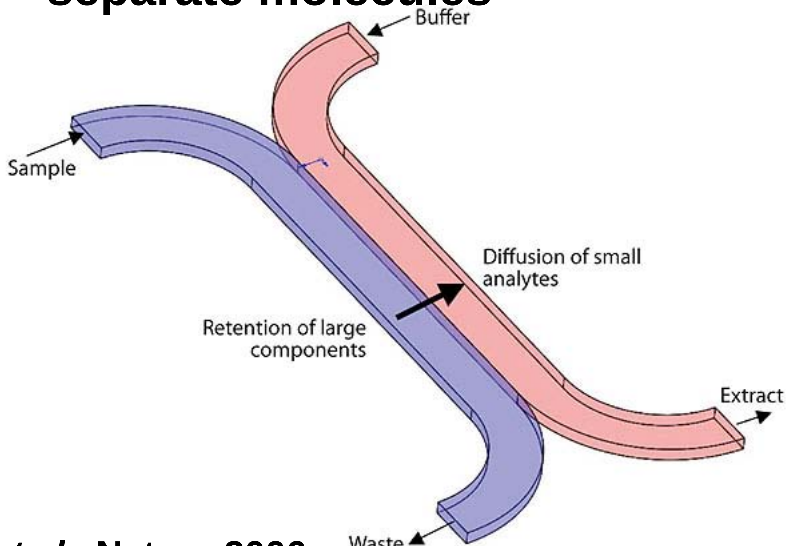
3-D cutaway of a micromachined microengine.
Photograph by Jonathan Protz.

Microfluidics

- This has been one of the most important domains of MEMS
  - Even though most microfluidics is not “MEMS”
  - And there are few commercial products
- The overall driver has been the life sciences
  - Though the *only* major commercial success is inkjets
- The initial driver was analytical chemistry
  - Separation of organic molecules
- More recently, this has shifted to biology
  - Manipulation of DNA, proteins, cells, tissues, etc.

Microfluidics examples

- > **H-filter**
  - **Developed by Yager and colleagues at UWash in mid-90's**
  - **Being commercialized by Micronics**
- > **An intrinsically microscale device**
  - **Uses diffusion in laminar flow to separate molecules**



Yager *et al.*, Nature 2006

Courtesy of Paul Yager, Thayne Edwards, Elain Fu, Kristen Helton, KjellNelson, Milton R. Tam, and Bernhard H. Weigl.
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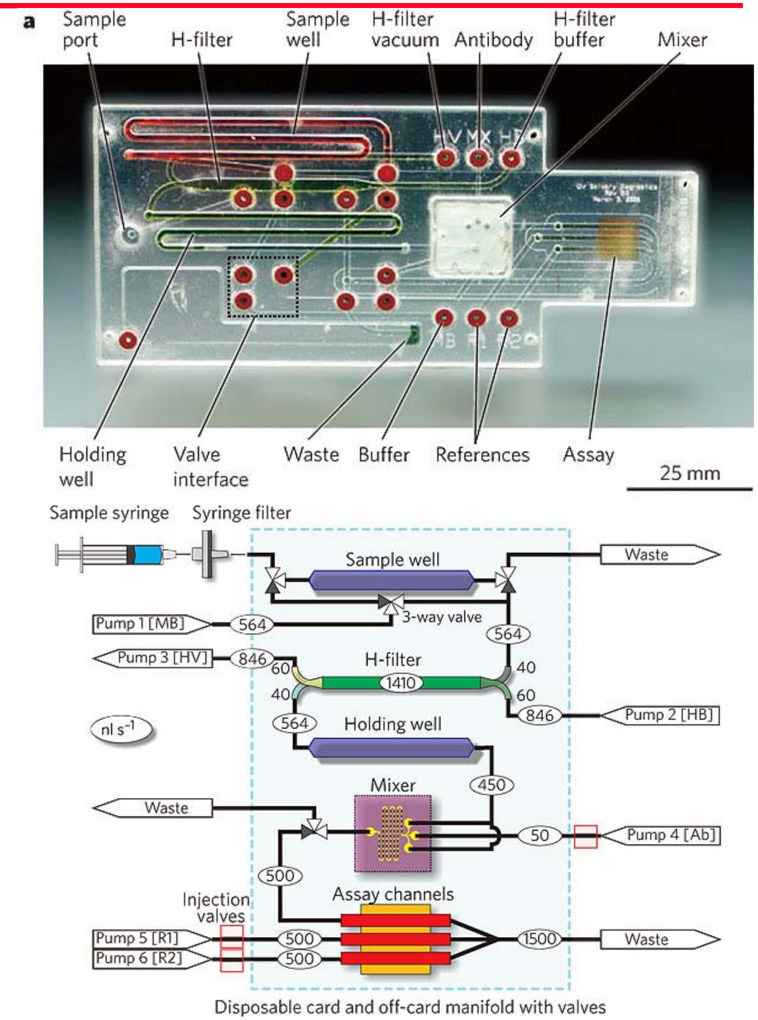

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Yager, P., T. Edwards, E. Fu, K. Helton, K. Nelson, M. R. Tam, and B. H. Weigl. "Microfluidic Diagnostic Technologies for Global Public Health." *Nature 442* (July 27, 2006): 412-418.

Microfluidics

> **Multi-layer elastomeric microfluidics (Quake, etc.)**
- **Use low modulus of silicone elastomers to create hydraulic valves**
- **Move liquids around**
- **Use diffusivity of gas in elastomer to enable dead-end filling**

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Please see: Figure 3 on p. 582 in: Thorsen, T., S. J. Maerkl, and S. R. Quake. "Microfluidic Large-Scale Integration." *Science, New Series* 298, no. 5593 (October 18, 2002): 580-584.

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Fluidigm

Goals

- To design microfluidics we need to understand
  - What pressures are needed for given flows
    - How do I size my channels?
  - What can fluids do at these scales
    - What are the relevant physics?
  - What things get better as we scale down
    - Mixing times, reagent volumes
  - What things get worse, and how can we manage them
    - Surface tension

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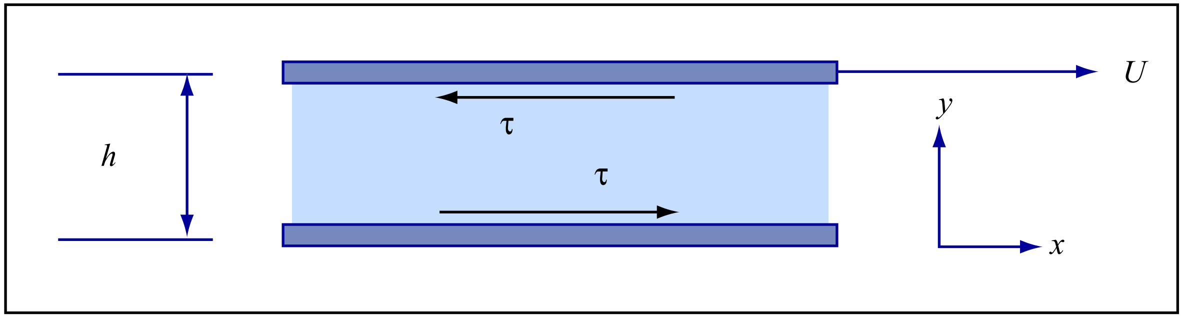
Viscosity

- When a solid experiences shear stress, it deforms (e.g., strains)
  - Shear Modulus relates the two
- When a fluid experiences shear stress, it deforms *continuously*
  - Viscosity relates the two
- Constitutive property describing relationship between shear stress [Pa] and shear rate [s^{-1}]
- Units: Pa-s
  - Water: 0.001 Pa-s
  - Air: 10^{-5} Pa-s



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Adapted from Figure 13.1 in Senturia, Stephen D. *Microsystem Design.* Boston, MA: Kluwer Academic Publishers, 2001, p. 318. ISBN: 9780792372462.

$$\tau = \eta \frac{U}{h}$$

and, in the differential limit

$$\tau = \eta \frac{\partial U_x}{\partial y}$$

A related quantity :

Kinematic Viscosity

$$\eta^* = \frac{\eta}{\rho_m}$$

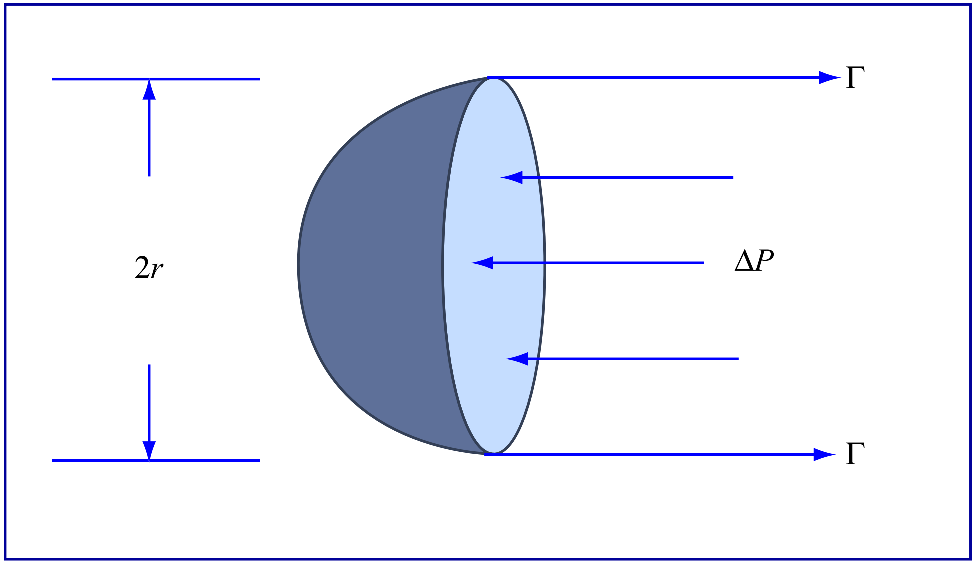
$$\tau = \eta^* \frac{\partial(\rho_m U_x)}{\partial y}$$

[m²/s]

This is a diffusivity for momentum

Surface Tension

- A liquid drop minimizes its free energy by minimizing its surface area. The effective force responsible for this is called **surface tension (Γ)** [J/m² = N/m]
- The surface tension creates a differential pressure on the two sides of a curved liquid surface



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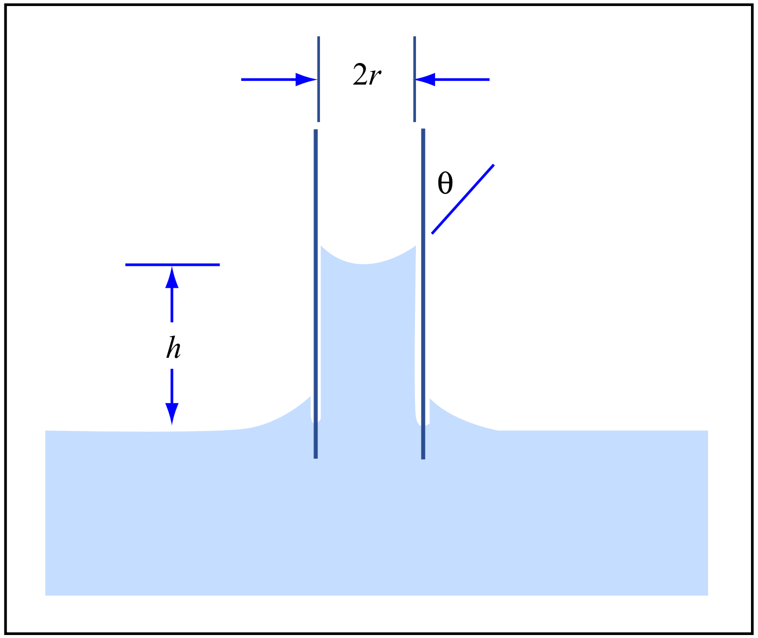
$$(2\pi r)\Gamma = \Delta P(\pi r^2)$$

solving for ΔP

$$\Delta P = \frac{2\Gamma}{r}$$

Capillary Effects

- **Surface forces can actually transport liquids**
- **Contact angles determine what happens, and these depend on the wetting properties of the liquid and the solid surface.**



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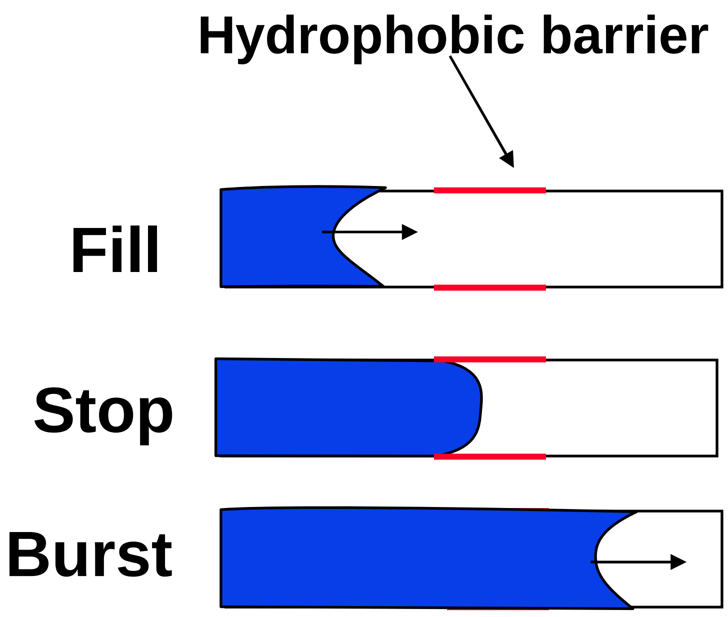
$$\rho_m g h\left(\pi r^2\right) = 2\pi r \Gamma \cos\theta$$

$$h = \frac{2\Gamma\cos\theta}{\rho_m g r}$$

Capillary Effects

- A hydrophobic valve



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Surface tension

> The scaling as 1/r is what makes dealing with surface tension HARD at the microscale

> Solutions

- Prime with low-surface tension liquids
  - » Methanol (Γ=22.6 mN/m) vs. water (Γ=72.8 mN/m)
  - » Or use surfactants
- Use CO_2 instead of air
  - » Dissolves more readily in water
  - » Zengerle *et al.,* IEEE MEMS 1995, p340
- Use diffusivity of gas in PDMS

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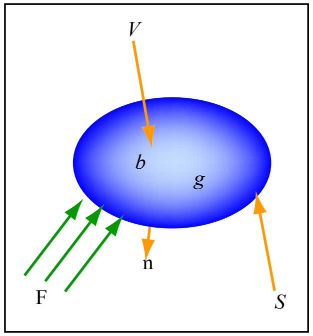
Fluid Mechanics Governing Equations

- Mass is conserved → Continuity equation
- Momentum is conserved → Navier-Stokes equation
- Energy is conserved → Euler equation
- We consider only the first two in this lecture

Continuity equation

- Conservation of mass
- In this case, for a control or "fixed" fluid volume
  - both S and V are constant in time
- Point conservation relation is valid for fixed or moving point



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$$\frac{d}{dt}\int b dV = -\int \mathbf{F}\cdot\mathbf{n} dS + \int g dV$$

$$\frac{\partial b}{\partial t} = -\nabla\cdot\mathbf{F} + g$$

$$m = \int_{volume} \rho_m dV$$

$$\frac{d}{dt}\int_{volume} \rho_m dV = -\int_{surface} \rho_m \mathbf{U}\cdot\mathbf{n}\ dS$$

$$\int_{volume} \frac{\partial \rho_m}{\partial t} dV + \int_{surface} \rho_m \mathbf{U}\cdot\mathbf{n}\ dS = 0$$

Apply the divergence theorem:

$$\int_{volume}\left[\frac{\partial \rho_m}{\partial t} + \nabla\cdot\left(\rho_m \mathbf{U}\right)\right] dV = 0$$

which implies

$$\boxed{\frac{\partial \rho_m}{\partial t} + \nabla\cdot\left(\rho_m \mathbf{U}\right) = 0}$$

$$\frac{\partial \rho_m}{\partial t} + \mathbf{U}\cdot\nabla\rho_m + \rho_m \nabla\cdot\mathbf{U} = 0$$

Material Derivative

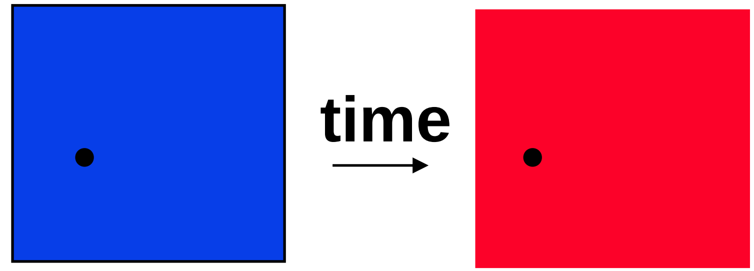
> The density can change due to three effects:

- An explicit time dependence (e.g. local heating)
- Flow carrying fluid through changing density regions
- Divergence of the fluid velocity

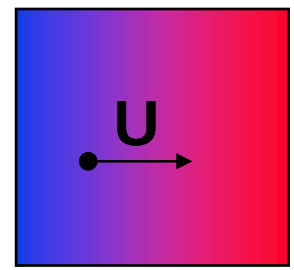
> The first two of these are grouped into the "material derivative"

- Rate of change for an observer moving with the fluid

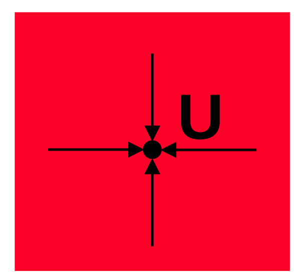
$$\frac{\partial \rho_m}{\partial t}$$



$$\mathbf{U} \cdot \nabla \rho_m$$



$$\rho_m (\nabla \cdot \mathbf{U})$$



Material Derivative

- **The first two of these are grouped into the "material derivative"**

$$\frac{\partial \rho_m}{\partial t} + \mathbf{U} \cdot \nabla \rho_m + \rho_m \nabla \cdot \mathbf{U} = 0$$

If we define

$$\frac{D\rho_m}{Dt} = \frac{\partial \rho_m}{\partial t} + \mathbf{U} \cdot \nabla \rho_m$$

or. more generally, define the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$$

we can write the continuity equation as

$$\frac{D\rho_m}{Dt} + \rho_m \nabla \cdot \mathbf{U} = 0$$

Incompressible

If the density is uniform, then in steady state

$$\nabla \cdot \mathbf{U} = 0$$

Momentum Conservation

- We want to write Newton's 2nd Law for fluids
- We start with a volume that travels with the fluid ➔ contains a constant amount of mass
  - Material volume
- Then go to an arbitrary control volume
  - Must account for flux of momentum through surface

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{U})}{dt}$$

$$\Downarrow$$

$$\mathbf{F} = \frac{d}{dt}\int_{V_m(t)} \rho_m \mathbf{U}\, dV_m$$

$$\Downarrow$$

velocity of surface

$$\mathbf{F} = \frac{d}{dt}\int_{V(t)} \rho_m \mathbf{U}\, dV + \int_{S(t)} \rho_m \mathbf{U}\left[\left(\mathbf{U} - \mathbf{U}_s\right)\cdot \mathbf{n}\right] dS$$

flux of $\rho_m\mathbf{U}$ thru surface

Momentum Conservation

- Pull time derivative into integral
- Cancel terms
- Apply divergence theorem

$$\mathbf{F} = \frac{d}{dt}\int_{V(t)} \rho_m \mathbf{U} dV + \int_{S(t)} \rho_m \mathbf{U}\left[\left(\mathbf{U}-\mathbf{U}_s\right)\cdot\mathbf{n}\right] dS$$

$$= \boxed{\int_{V(t)} \frac{\partial}{\partial t}\rho_m \mathbf{U} dV + \int_{S(t)} \rho_m \mathbf{U}\left[\cancel{\mathbf{U}_s}\cdot\mathbf{n}\right] dS} + \int_{S(t)} \rho_m \mathbf{U}\left[\left(\mathbf{U}-\cancel{\mathbf{U}_s}\right)\cdot\mathbf{n}\right] dS$$

Leibniz's rule

$$= \int_{V(t)} \frac{\partial}{\partial t}\rho_m \mathbf{U} dV + \int_{S(t)} \rho_m \mathbf{U}\mathbf{U}\cdot\mathbf{n} dS$$

$$\Downarrow \int_{S(t)} \mathbf{a}\cdot\mathbf{n} dS = \int_{V(t)} \nabla\cdot\mathbf{a} dV$$

Momentum Conservation

- Final result does not depend on control volume

$$= \int_{V(t)} \frac{\partial}{\partial t} \rho_m \mathbf{U} dV + \int_{V(t)} \nabla \cdot \rho_m \mathbf{U}\mathbf{U} dV$$

$$= \int_{V(t)} \left[\mathbf{U} \cancel{\left(\frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m \mathbf{U} \right)} + \rho_m \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) \right] dV$$

$$\Downarrow$$

$$\boxed{\mathbf{F} = \int_{V} \rho_m \frac{D\mathbf{U}}{Dt} dV}$$

Momentum Conservation

- Add in force terms
- Include body forces (e.g., gravity)
- And surface forces (i.e., stresses)

$$\mathbf{F}_b = \int_V \rho_m \mathbf{g}\, dV$$

$$\mathbf{F}_s = \int_{S(t)} \mathbf{s}(\mathbf{n}) dS$$

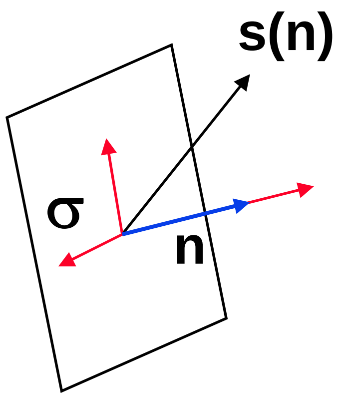
$$\mathbf{s}(\mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\sigma}$$

$$\mathbf{F}_s = \int_{S(t)} \mathbf{n} \cdot \boldsymbol{\sigma} dS$$

$$\mathbf{F}_s = \int_{V(t)} \nabla \cdot \boldsymbol{\sigma} dV$$

$$\Downarrow$$

$$\rho_\mathrm{m} \frac{D\mathbf{U}}{Dt} = \rho_m \mathbf{g} + \nabla \cdot \boldsymbol{\sigma}$$



Navier-Stokes Equations

- Substitute in for stress tensor

$$\boldsymbol{\sigma} = -P\mathbf{n} + \boldsymbol{\tau}$$

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla P + \nabla \cdot \boldsymbol{\tau}$$

- Compressible Newtonian fluid constitutive relation

$$\nabla \cdot \boldsymbol{\tau}$$

$$\Downarrow \qquad \tau = \eta \frac{\partial U_x}{\partial y}$$

$$\eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla \left(\nabla \cdot \mathbf{U} \right)$$

$$\Downarrow$$

- Compressible Navier-Stokes equations

$$\rho_{\mathrm{m}} \frac{D\mathbf{U}}{Dt} = -\nabla P + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla \left(\nabla \cdot \mathbf{U} \right) + \rho_m \mathbf{g}$$

$$P^* = P - \rho_m \mathbf{g} \cdot \mathbf{r}$$

$$\rho_{\mathrm{m}} \frac{D\mathbf{U}}{Dt} = -\nabla P^* + \eta \nabla^2 \mathbf{U} + \frac{\eta}{3} \nabla \left(\nabla \cdot \mathbf{U} \right)$$

Navier-Stokes Equations

- Terms in compressible N-S equations

time-dependence

pressure

compressibility

$$\rho_{\mathrm{m}}\left(\frac{\partial \mathbf{U}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{U}\right)=-\nabla P+\eta\nabla^{2}\mathbf{U}+\frac{\eta}{3}\nabla\left(\nabla\cdot\mathbf{U}\right)+\rho_{m}\mathbf{g}$$

inertial

viscous stresses

gravity

Dimensionless Numbers

- Fluid mechanics is full of non-dimensional numbers that help classify the types of flow
- Reynolds number is most important
- Reynolds number:
  - The ratio of inertial to viscous effects
  - Ratio of convective to diffusive momentum transport
  - Small Reynolds number means neglect of inertia
  - Flow at low Reynolds number is laminar

$$\mathrm{Re}\left(\tilde{\mathbf{U}}\cdot\tilde{\nabla}\tilde{\mathbf{U}}\right) = -(A)\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2\tilde{\mathbf{U}}$$

Non-dimensionalized steady incompressible flow

$$Re = \frac{\rho_m L_0 U_0}{\eta} = \frac{L_0 U_0}{\eta^*} = \frac{U_0}{\eta^* / L_0}$$

See Deen, *Analysis of Transport Phenomena*

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Incompressible Laminar Flow

- **The Navier-Stokes equation becomes very "heat-flow-equation-like," although the presence of DU/Dt instead of ∂U/∂t makes the equation nonlinear, hence HARD**

Navier-Stokes becomes

$$\rho_{\mathrm{m}} \frac{D\mathbf{U}}{Dt} = -\nabla P^* + \eta \nabla^2 \mathbf{U}$$

to obtain a "diffusion-like" equation:

$$\rho_{\mathrm{m}} \frac{D\mathbf{U}}{Dt} = \eta \nabla^2 \mathbf{U} - \nabla P^*$$

As aside on heat convection

- Our heat-flow equation looked like
- Compare to incompressible N-S eqn
- If we allow fluid to move—to convect—we can include convection in our heat conservation
- At steady state, we get a relation that allows us to compare convective heat transport to conduction
- This is the Peclet number
- For microscale water flows, L~100 μm, U~0.1 mm/s, D~150x10^{-6} m^2/s

$$\frac{\partial T}{\partial t} = D\nabla^2 T + \frac{1}{\cancelto{0}{C_0}}\cancelto{0}{P_0}\Big|_{sources}$$

$$\rho_\mathrm{m}\frac{D\mathbf{U}}{Dt} = \eta\nabla^2\mathbf{U} - \nabla P^*$$

$$\frac{DT}{Dt} = D\nabla^2 T + \frac{1}{\cancelto{0}{C_0}}\cancelto{0}{P_0}\Big|_{sources}$$

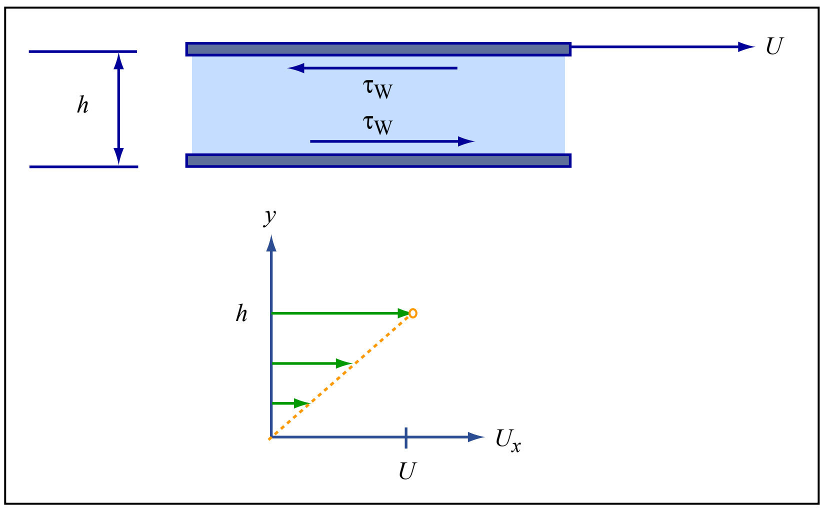
$$\mathbf{U}\cdot\nabla T = D\nabla^2 T + \frac{1}{\cancelto{0}{C_0}}\cancelto{0}{P_0}\Big|_{sources}$$

$$Pe = \frac{LU}{D} = \frac{\left(10^{-4}\,m\right)\left(10^{-4}\,m/s\right)}{0.15\cdot 10^{-6}\,m^2/s} \sim 0.1$$

Couette or Shear Flow

- Pure shear flow with a linear velocity profile
- No pressure gradient

- Relative velocity goes to zero at the walls (the so-called no-slip boundary condition)



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Boston, MA: Kluwer Academic Publishers, 2001, p. 327. ISBN: 9780792372462.

The flow is one-dimensional

$$\mathbf{U} = U_x(y)\hat{\mathbf{n}}_x$$

$$\rho_{\mathrm{m}}\left(\cancel{\frac{d\mathbf{U}}{dt}} + \cancel{\mathbf{U}\cdot\nabla\mathbf{U}}\right) = \eta\nabla^2\mathbf{U} - \cancel{\nabla P^*}$$

N-S Eqns collapse to the Laplace eqn

$$\frac{\partial^2 U_x}{\partial y^2} = 0$$

$$U_x = c_1 y + c_2$$

$$\text{B.C.'s:}\quad U_x(0) = 0, U_x(h) = U$$

$$\Downarrow$$

$$U_x = \frac{y}{h}U$$

Poiseuille Flow

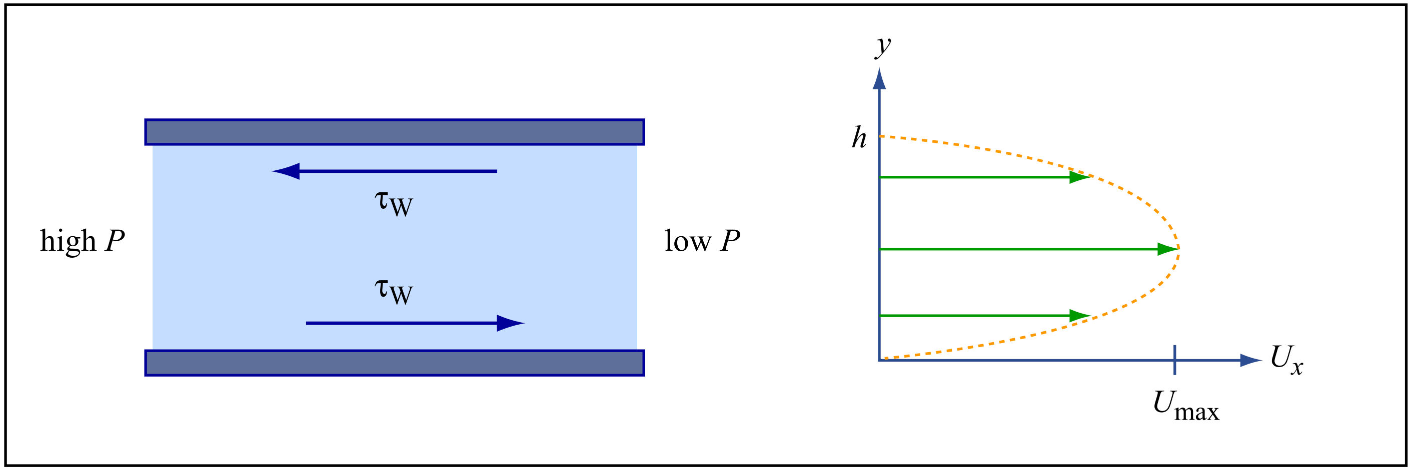
> **Pressure-driven flow through a pipe**

- **In our case, two parallel plates**

> **Velocity profile is parabolic**

> **This is the most common flow in microfluidics**

- **Assumes that h<<W**



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Solution for Poiseuille Flow

- Assume a uniform pressure gradient along the pipe

$$\frac{dP}{dx} = -K$$

- Assume x-velocity only depends on y

$$\frac{\partial^2 U_x}{\partial y^2} = -\frac{K}{\eta}$$

- Enforce zero-flow boundary conditions at walls

$$U_x = \frac{1}{2\eta}\left[y\left(h-y\right)\right]K$$

- Maximum velocity is at center

$$U_{\max} = \frac{h^2}{8\eta}K$$

- Volumetric flowrate is

$$Q = W\int_0^h U_x dy = \frac{Wh^3}{12\eta}K$$

Lumped Model for Poiseuille Flow

- **Can get lumped resistor using the fluidic convention**
- **Note STRONG dependence on *h***
- **This relation is more complicated when the aspect ratio is not very high…**

$$\Delta P = \text{effort} = KL$$

$$\Delta P = \frac{12\eta L}{Wh^3} Q$$

$$\Rightarrow R_{Pois} = \frac{12\eta L}{Wh^3}$$

Development Length

- It takes a certain characteristic length, called the **development length**, to establish the Poiseuille velocity profile
- This development length corresponds to a development time for viscous stresses to diffuse from wall
- Development length is proportional to the characteristic length scale and to the Reynolds number, both of which tend to be small in microfluidic devices

$$time \approx L^2 / \eta^* \approx \mathrm{Re}\frac{L}{U}$$

$$L_D \approx (time)U \approx \mathrm{Re}\, L$$

A note on vorticity

- A common statement is to say that laminar flow has no vorticity
- What is meant is that laminar flow has no turbulence
- Vorticity and turbulence are different
- Can the pinwheel spin?
  - Then there is vorticity
- Demonstrate for Poiseuille flow

$$U_x = \frac{1}{2\eta}\left[y(h-y)\right]K$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{U}$$

$$\boldsymbol{\omega} = \mathbf{n}_\mathbf{y}\frac{\partial U_x}{\partial z} - \mathbf{n}_z\frac{\partial U_x}{\partial y}$$

$$\boldsymbol{\omega} = -\mathbf{n}_z\frac{K}{2\eta}(h-2y)$$