Special Topics in Structures: Residual Stress and Energy Methods

Carol Livermore

Massachusetts Institute of Technology

*** With thanks to Steve Senturia, from whose lecture notes some of these materials are adapted.**

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Outline

- Effects of residual stresses on structures
- Energy methods
  - Elastic energy
  - Principle of virtual work: variational methods
  - Examples
- Rayleigh-Ritz methods for resonant frequencies and extracting lumped-element masses for structures

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Reminder: Thin Film Stress

- If a thin film is adhered to a substrate, mismatch of thermal expansion coefficient between film and substrate can lead to stresses in the film (and, to a lesser degree, stresses in the substrate)
- Residual stress can also come from film structure: intrinsic stress
- Stresses set up bending moments that can bend the substrate
- When we release a residually stressed MEMS structure, interesting effects can ensue

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Reminder: Differential equation of beam bending

- Small angle bending:

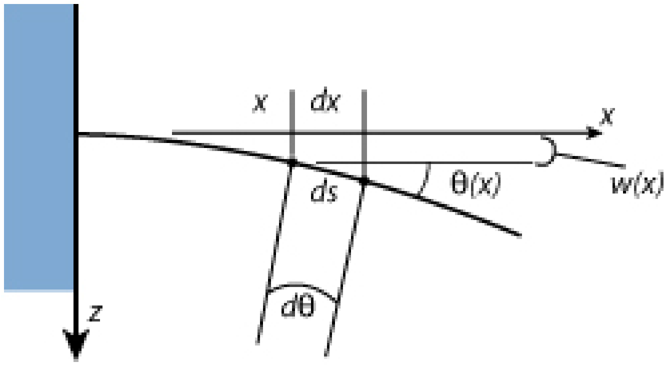


Image by MIT OpenCourseWare.
Adapted from Figure 9.11 in: Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, pp. 214.
ISBN: 9780792372462.

- Beam equation:
  - **q = distributed load**
  - **w = vertical displacement**
  - **x = axial position along beam**

$$\frac{d^4 w}{dx^4} = \frac{q}{EI}$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Fixed-fixed beam

- Fixed-fixed beams are common in MEMS: switches, diffraction gratings, flexures
- Example: Silicon Light Machines Grating Light Valve display deflects a beam in order to diffract light

Image removed due to copyright restrictions.

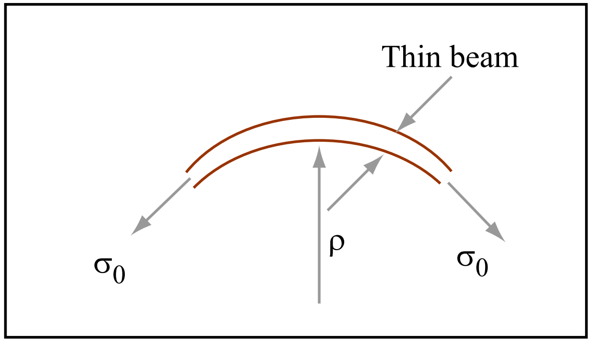
Please see: Figure 1.4 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 7. ISBN: 9780792372462.

- Residual stress in beams can enhance or reduce response to an applied load, and impact flatness of actuated beam
- Residual stress can be included in the basic beam bending equation by the addition of an extra term

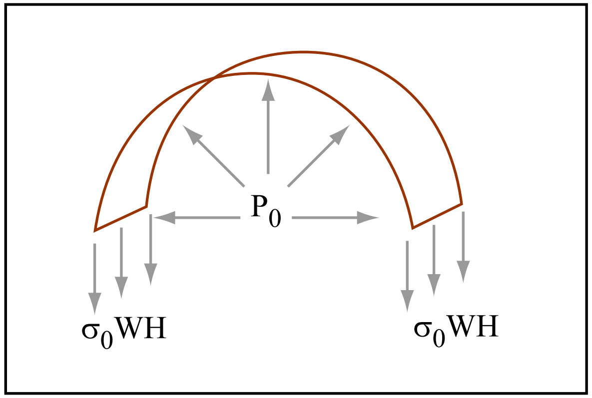
Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Residual Axial Stress in Beams

- Residual axial stress in a beam contributes to its bending stiffness
- Leads to the Euler beam equation



Images by MIT OpenCourseWare.
Adapted from Figure 9.15 in Senturia, Stephen D. *Microsystem Design*.
Boston, MA: Kluwer Academic Publishers, 2001, p. 227. ISBN: 9780792372462.



Images by MIT OpenCourseWare.
Adapted from Figure 9.16 in Senturia, Stephen D. *Microsystem Design*.
Boston, MA: Kluwer Academic Publishers, 2001, p. 228. ISBN: 9780792372462.

$$2\rho W P_0 = 2\sigma_0 W H$$

$$\Rightarrow P_0 = \frac{\sigma_0 H}{\rho}$$

which is equivalent to a distributed load

$$q_0 = P_0 W = \sigma_0 W H \frac{d^2 w}{dx^2}$$

Insert as added load into beam equation :

$$EI \frac{d^4 w}{dx^4} = q + q_0$$

$$\boxed{EI \frac{d^4 w}{dx^4} - \sigma_0 W H \frac{d^2 w}{dx^2} = q}$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Effect of tensile stress on stiffness

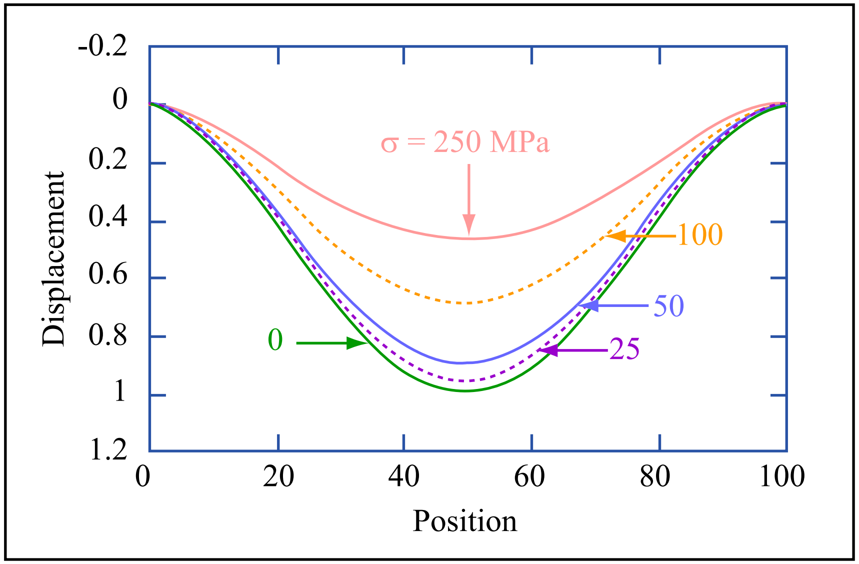


Image by MIT OpenCourseWare.

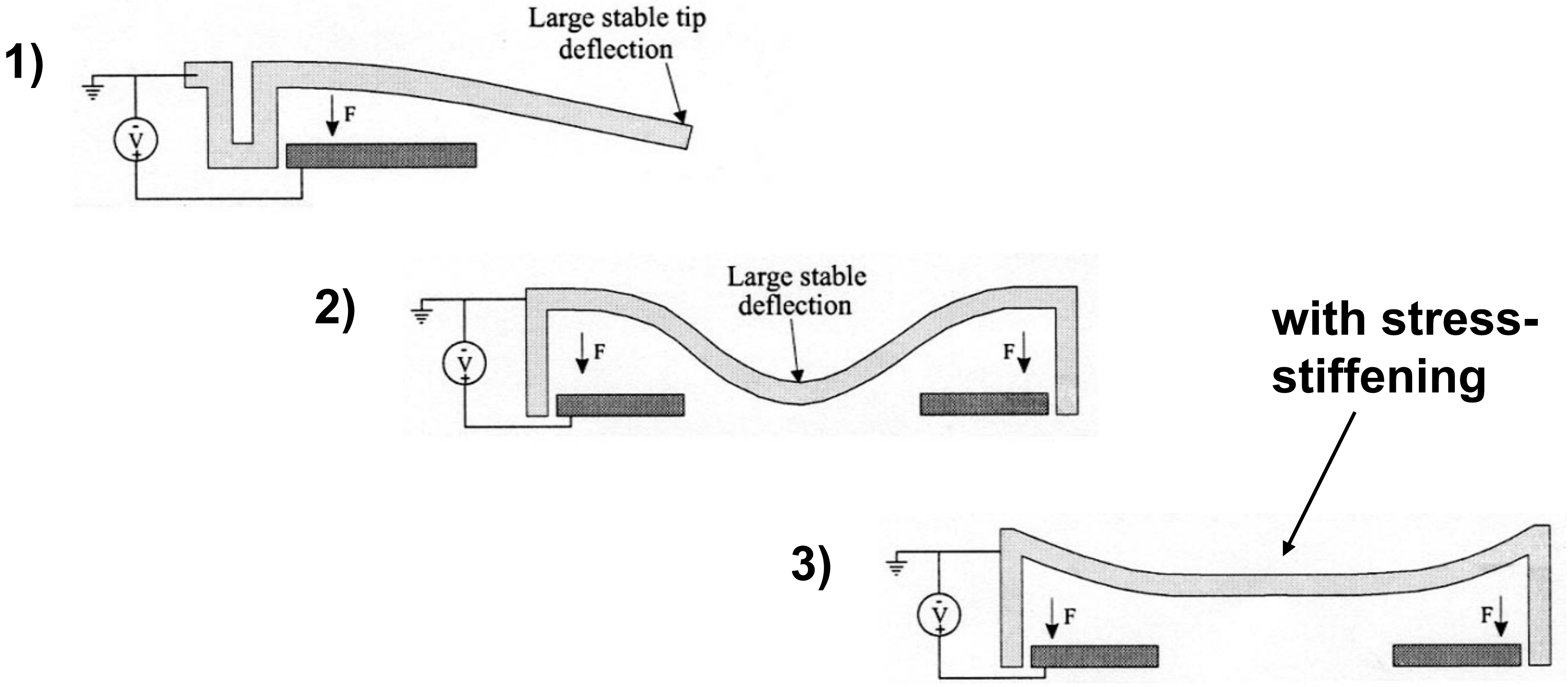
Adapted from Figure 9.17 in Senturia, Stephen D. *Microsystem Design.*
Boston, MA: Kluwer Academic Publishers, 2001, p. 232. ISBN: 9780792372462.

100 μm long, 2 μm wide, 2 μm high fixed-fixed silicon beam

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Stress impacts flatness: leveraged bending

- Pull-in is modified if the actuating electrodes are away from the point of closest approach



with stress-stiffening

Figure 3 on p. 499 in: Hung, E. S., and S. D. Senturia. "Extending the Travel Range of Analog-tuned Electrostatic Actuators." *Journal of Microelectromechanical Systems* 8, no. 4 (December 1999): 497-505. © 1999 IEEE.

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Buckling of Axially Loaded Beams

- If the compressive stress is too large, a beam will spontaneously bend – this is called *buckling*
- The basic theory of buckling is in Sec. 9.6.3
- The Euler buckling criterion:

$$\sigma_{Euler} = -\frac{\pi^2}{3}\frac{EH^2}{L^2}$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Plates with in-plane stress and membranes

- As with the Euler beam equation, in plane stress can be included

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-\left(\frac{N_x}{W}\frac{\partial^2 w}{\partial x^2}+\frac{N_y}{W}\frac{\partial^2 w}{\partial y^2}\right)=P(x,y)$$

Axial stresses in x and y directions

- When tensile stress dominates over flexural rigidity (thin, tensioned plate), the plate may be considered a membrane

$$\left(\frac{N_x}{W}\frac{\partial^2 w}{\partial x^2}+\frac{N_y}{W}\frac{\partial^2 w}{\partial y^2}\right)=-P$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

How about cantilevers?

- Example: residually stressed cantilever, where stress is constant throughout structure
- Before release: stressed cantilever is attached to surface
- After release: cantilever relieves stress by expanding or contracting to its desired length
- No bending of released structure

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

How about nonuniform axial stress?

- Nonuniform axial stress through the thickness of a beam creates a bending moment
- It can arise from two sources
  - Intrinsic stress gradients, created during formation of the cantilever material (e.g. polysilicon)
  - Residual stress in thin films deposited onto the cantilever
- The bending moment curls the cantilever

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Cantilever with stress gradient

> **Think about it in three steps:**

- **Relax the average stress to zero after release**
- **Compute the moment when the beam is flat**
- **Compute the curvature that results from the moment**

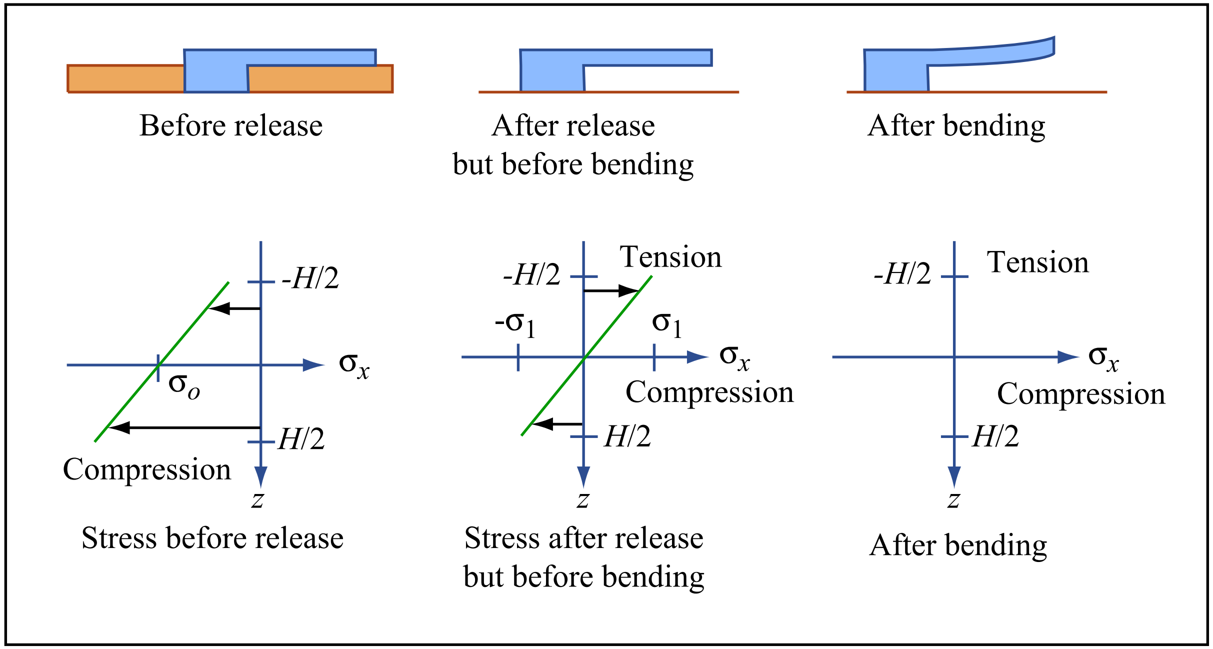


Image by MIT OpenCourseWare.

Adapted from Figure 9.13 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 223. ISBN: 9780792372462.

$$M_x = \int_A z\sigma_x dA = -\frac{1}{6}WH^2\sigma_1 \quad \text{and} \quad \rho_x = -\frac{EI}{M_x} \Rightarrow \rho_x = \frac{1}{2}\frac{EH}{\sigma_1}$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Thin Film on Cantilever

- In this case, the curling does not relieve all the stress

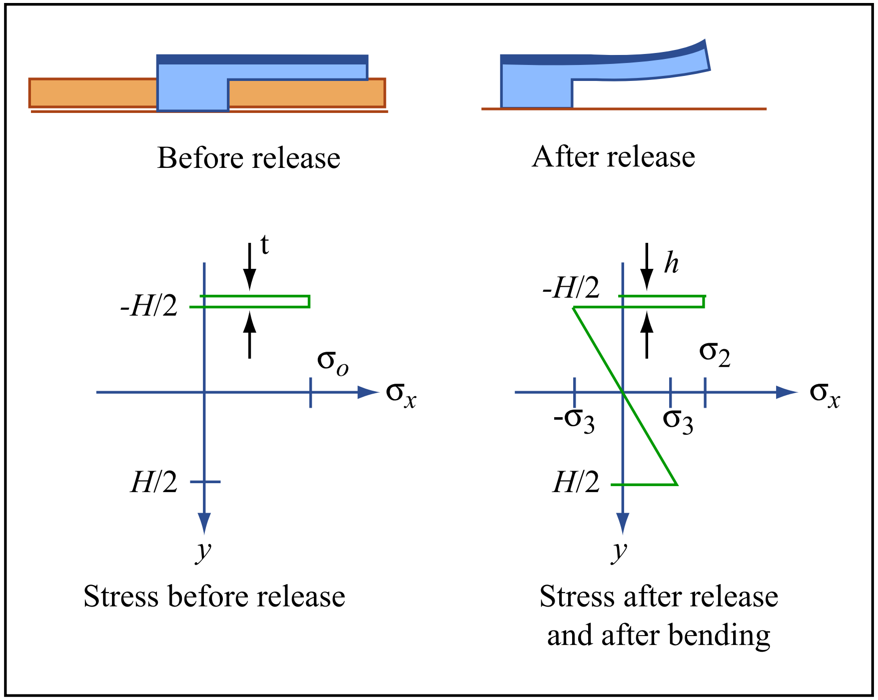
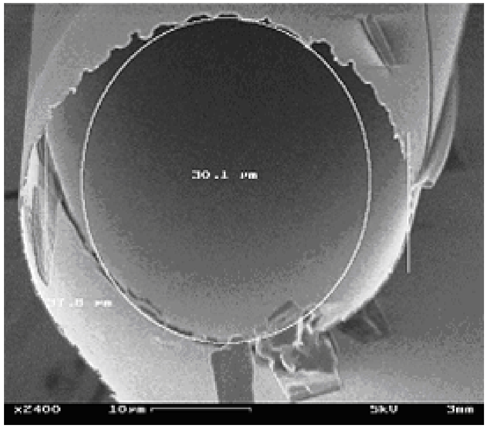


Image by MIT OpenCourseWare.

Adapted from Figure 9.14 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 224. ISBN: 9780792372462.



Barbastathis group, MIT

Courtesy of George Barbastathis. Used with permission.

- See text for math

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Outline

- Effects of residual stresses on structures
- Energy methods
  - Elastic energy
  - Principle of virtual work: variational methods
  - Examples
- Rayleigh-Ritz methods for resonant frequencies and extracting lumped-element masses for structures

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Elastic Energy

- Elastic stored energy density is the integral of stress with respect to strain

$$\text{Elastic energy density:} \quad \widetilde{W}(x,y,z) = \int_0^{\varepsilon(x,y,z)} \sigma(\varepsilon)d\varepsilon$$

$$\text{When } \sigma(\varepsilon) = E\varepsilon: \quad \widetilde{W}(x,y,z) = \frac{1}{2}E\left[\varepsilon(x,y,z)\right]^2$$

- The total elastic stored energy is the volume integral of the elastic energy density

$$\text{Total stored elastic energy:} \quad W = \iiint_{Volume} \widetilde{W}(x,y,z)dxdydz$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Including Shear Strains

- More generally, the energy density in a linear elastic medium is related to the product of stress and strain
- A similar approach can be used for electrostatic stored energy density (1/2)D*E and magnetostatic stored energy density (1/2)B*H.

$$\text{For axial strains:} \quad \tilde{W} = \frac{1}{2}\sigma\varepsilon$$

$$\text{For shear strains:} \quad \tilde{W} = \frac{1}{2}\tau\gamma$$

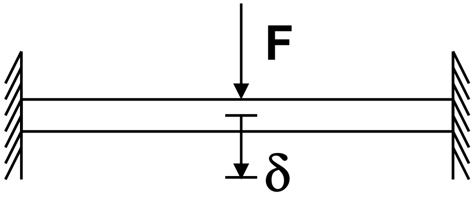
$$\text{This leads to a total elastic strain energy:}$$

$$W = \frac{1}{2}\iiint_{Volume}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)dxdydz$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Concept: Principle of Virtual Work

- The question: how to determine the deformation that results from an applied load



$\delta(F) = ??$

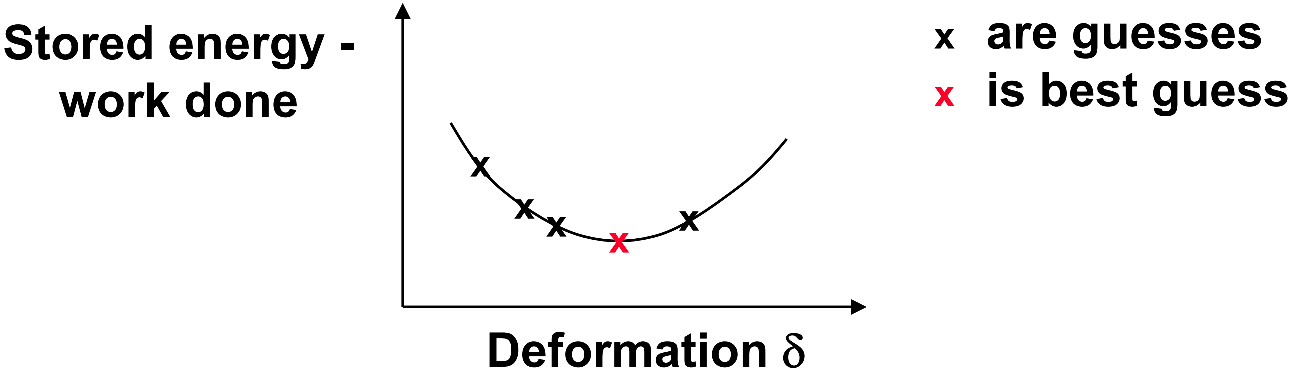
- Known: the work done on an energy-conserving system by external forces must result in an equal amount of stored potential energy

- Imposing this condition can provide an exact solution to many problems
  - For example, if functional dependence between quantities is known, and you just need to find what the actual values are

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Concept: Principle of Virtual Work

- Can approach this from a "guessing" point of view
  - Guess values for δ; whichever one best equates stored energy and work done is the right answer



- What if you don't know the functional form of your deformations/displacements – does this still work?
- Yes! You can choose a plausible shape function for the displacement with a few adjustable parameters and iteratively "guess" the constants to best equate stored energy and work done

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Principle of Virtual Work

- Goal: a variational method for solving energy-conserving problems (a mathematical way of approaching the "guessing")
- Define total potential U, including work and stored energy

$$U = \text{Stored energy - Work done}$$

- A system in equilibrium has a total potential U that is a minimum with respect to any virtual displacement
  - No matter what you change, you won't get any closer to matching work and stored energy
- Requirement: the virtual displacement must obey B.C.
- Nomenclature for small virtual displacements
  - In the x direction: δu
  - In the y direction: δv
  - In the z direction: δw

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Math: Principle of Virtual Work

- Consider all possible virtual displacements; evaluate change in strains

$$\delta\varepsilon_x = \frac{\partial}{\partial x}\delta u \quad \text{and} \quad \delta\gamma_{xy} = \left(\frac{\partial}{\partial x}\delta v + \frac{\partial}{\partial y}\delta u\right)$$

- This implies changes in strain energy density

$$\delta\widetilde{W} = \sigma_x\delta\varepsilon_x + \ldots + \tau_{xy}\delta\gamma_{xy} + \ldots$$

- The principle of virtual work states that in equilibrium, for any virtual displacement that is compatible with the B.C.,

$$\iiint\limits_{Volume} \delta\widetilde{W}dxdydz - \iint\limits_{Surface}\left(F_{s,x}\delta u + F_{s,y}\delta v + F_{s,z}\delta w\right)dS$$

$$- \iiint\limits_{Volume}\left(F_{b,x}\delta u + F_{b,y}\delta v + F_{b,z}\delta w\right)dxdydz = 0$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Differential Version

- The previous equation is equivalent to the following:

$$\delta\left[\underset{Volume}{\iiint}\widetilde{W}dxdydz-\underset{Surface}{\iint}\left(F_{s,x}u+F_{s,y}v+F_{s,z}w\right)dS-\underset{Volume}{\iiint}\left(F_{b,x}u+F_{b,y}v+F_{b,z}w\right)dxdydz\right]=0$$

This can be restated in the following form :

$$\delta U=0$$

where

$$U=\left[\underset{Volume}{\iiint}\widetilde{W}dxdydz-\underset{Surface}{\iint}\left(F_{s,x}u+F_{s,y}v+F_{s,z}w\right)dS-\underset{Volume}{\iiint}\left(F_{b,x}u+F_{b,y}v+F_{b,z}w\right)dxdydz\right]$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Variational methods

> Select a trial solution with parameters that can be varied
  - $\hat{u}(x, y, z; c_1, c_2, \ldots c_n)$ = trial displacement in x
  - $v(x, y, z; c_1, c_2, \ldots c_n)$ = trial displacement in y
  - $\hat{w}(x, y, z; c_1, c_2, \ldots c_n)$ = trial displacement in z

> Formulate the total potential U of the system as functions of these parameters

> Find the potential minimum with respect to the values of the parameters

$$\frac{\partial U}{\partial c_1} = 0, \quad \frac{\partial U}{\partial c_2} = 0, \ldots . \quad \frac{\partial U}{\partial c_n} = 0$$

> The result is the best solution possible with the assumed trial function

Why Bother?

- Nonlinear partial differential equations are basically very nasty.
- Approximate analytical solutions can always be found with variational methods
- The analytical solutions have the correct dependence on geometry and material properties, hence, serve as the basis for good macro-models
- Accurate numerical answers may require finite-element modeling

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Analytic vs. Numerical

- **Analytic variational methods and numerical finite-element methods both depend on the Principal of Virtual Work**
- **Both methods minimize total potential energy**
- **FEM methods use local trial functions (one per element). Variational parameters are the nodal displacements**
- **Analytic methods use global trial functions**

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Outline

- Effects of residual stresses on structures
- Energy methods
  - Elastic energy
  - Principle of virtual work: variational methods
  - Examples
- Rayleigh-Ritz methods for resonant frequencies and extracting lumped-element masses for structures

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: fixed-fixed beam, small deflections

- **Doubly-fixed beam with a point load at some position along the beam, in the small deflection limit**

- **Our present choice: use a fourth degree polynomial trial solution**

$$\hat{w}(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

$$\text{Boundary conditions:} \quad w = 0 \ \text{ and } \ w' = 0 \ \text{ at } \ x = 0, L$$

- **Apply boundary conditions:**
  - **$c_0 = c_1 = 0$ from BC at x = 0**
  - **BC at x = L eliminate two more constants**
  - **Result is a shape function with one undetermined amplitude parameter**

$$\hat{w}(x) = c_4\left(L^2 x^2 - 2Lx^3 + x^4\right)$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: fixed-fixed beam, small deflections

- Formulate total potential energy and find the minimum
- Calculate strain energy from bending

$$\varepsilon = -\frac{z}{\rho} = -z\frac{d^2\hat{w}}{dx^2} = -zc_4\left(2L^2 - 12Lx + 12x^2\right)$$

width of beam

total strain energy

$$\mathrm{W} = \frac{EW}{2}\int_0^L \int_{-H/2}^{H/2} \varepsilon^2 dxdz = \frac{1}{30}EWH^3L^5c_4^2$$

- Calculate work done by external force applied at x_0

$$Work = F\hat{w}(x_0) = Fc_4\left(L^2x_0^2 - 2Lx_0^3 + x_0^4\right)$$

- This yields total potential energy

$$U = \frac{1}{30}EWH^3L^5c_4^2 - \left(L^2x_0^2 - 2Lx_0^3 + x_0^4\right)Fc_4$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: fixed-fixed beam, small deflections

- Minimize total potential energy with respect to c_4, determine c_4, and plug in to find variational solution for deflection w(x)

$$\frac{\partial U}{\partial c_4} = 0$$

$$c_4 = 15 \frac{L^2 x_0^2 - 2Lx_0^3 + x_0^4}{EWH^3 L^5} F$$

$$w = 15 \frac{\left(L^2 x_0^2 - 2Lx_0^3 + x_0^4\right)\left(L^2 x^2 - 2Lx^3 + x^4\right)}{EWH^3 L^5} F$$

- Compare stiffness for the case of a center-applied load

$$w\left(L/2\right) = \frac{15}{256} \frac{L^3}{EWH^3} F$$

$$k = \frac{256}{15} \frac{EWH^3}{L^3} \approx 17 \frac{EWH^3}{L^3}$$

Recall solution of beam equation

$$k = 16 \frac{EWH^3}{L^3}$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Properties of the Variational Solution

- Does it solve the beam equation? *NO*
- Is the point of maximum deflection near where the load is applied? *NOT IN GENERAL*
- How can we determine how accurate the solution is? *TRY A BETTER FUNCTION*
- Was this a good trial function? *NO*

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

A Better Trial Function

- Fifth-order polynomial allows both the amplitude and shape of the deformation to be varied

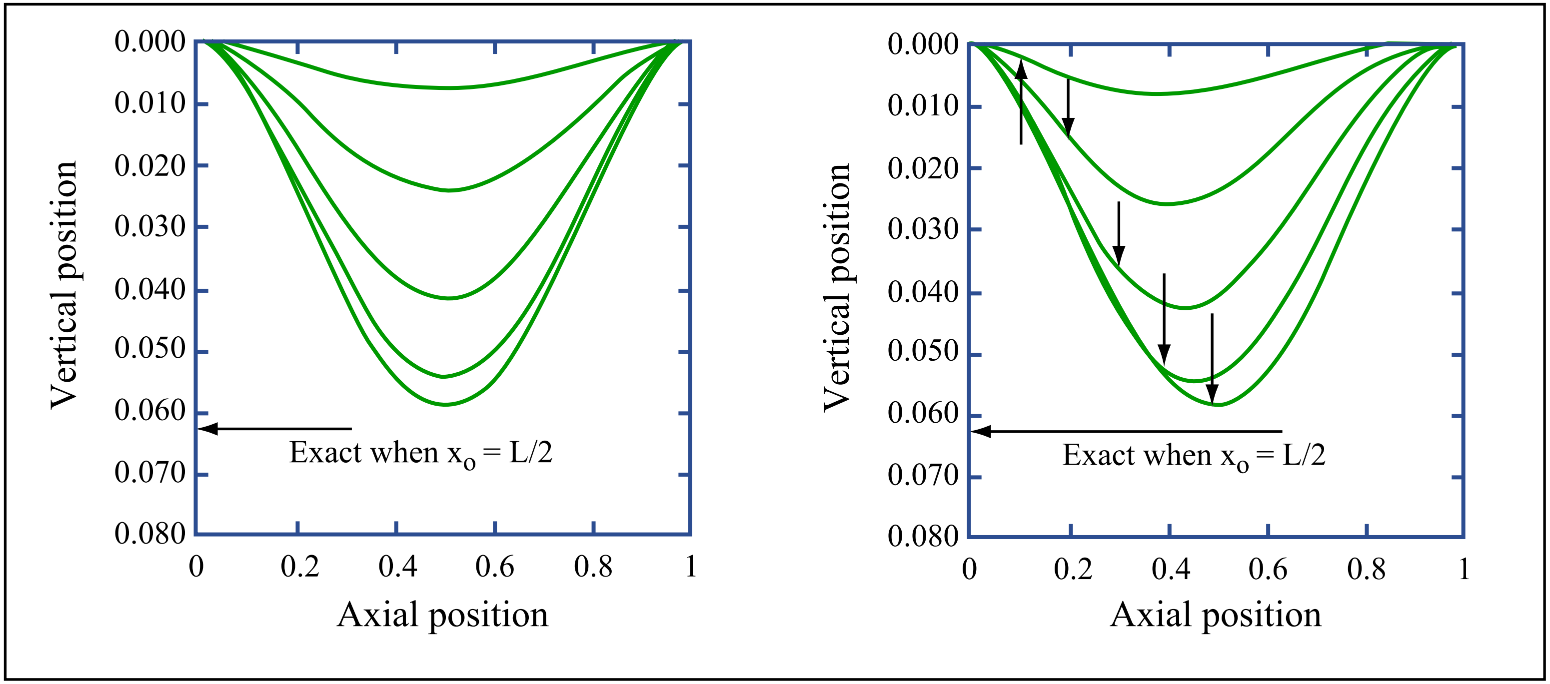


Image by MIT OpenCourseWare.

Adapted from Figure 10.1 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 248. ISBN: 9780792372462. The artist's representation of the fourth and fifth degree polynomials is approximate.

Fourth degree polynomial **Fifth degree polynomial**

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

What about large deflections?

- For small deflections, pure bending is a good approximation
  - The geometrically constructed neutral axis really does have about zero strain

- For large deflections, the beam gets longer
  - Tensile side gets even more tensile
  - Compressed side gets less compressed
  - Neutral axis becomes tensile

- We can treat this as a superposition of two events
  - First, the beam bends in pure bending, which draws the end of the beam away from the second support
  - Then, the beam is stretched to reconnect with the second support
  - Quantify the stretching by the strain at the originally neutral axis

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Analysis of “Large” Deflections

- When deflections are “large,” on the order of the beam thickness, stretching becomes important

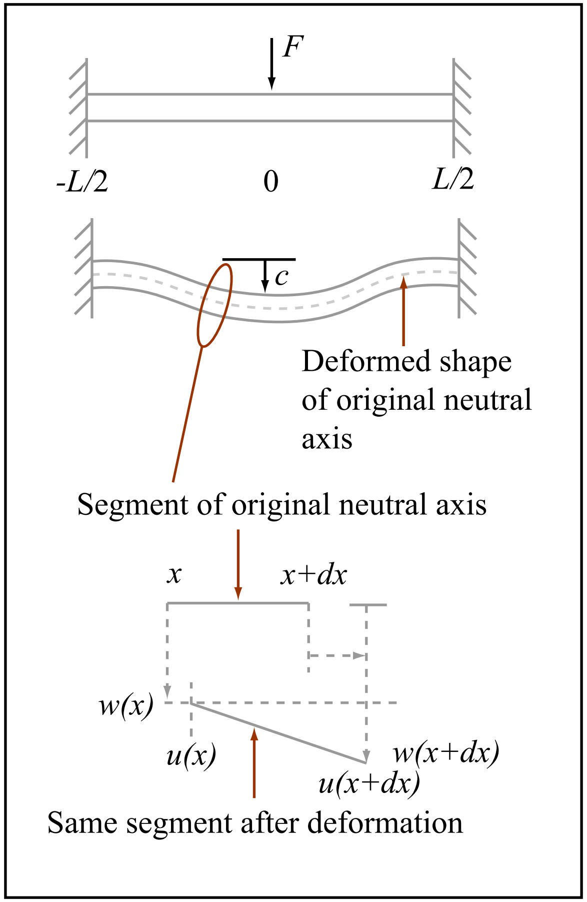


Image by MIT OpenCourseWare.

Adapted from Figure 10.2 in Senturia, Stephen D. *Microsystem Design*.
Boston, MA: Kluwer Academic Publishers, 2001, p. 250. ISBN: 9780792372462.

As a result of stretching, the arc length increases

$$ds = \sqrt{[dx + u(x+dx) - u(x)]^2 + [w(x+dx) - w(x)]^2}$$

Using the result that $\sqrt{1+\delta} = 1 + \frac{\delta}{2}$

$$ds = dx\left[1 + \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

The axial strain is given by

$$\varepsilon_x = \frac{ds - dx}{dx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

The change in length is

$$\delta L = \int_{-L/2}^{L/2} \frac{ds - dx}{dx} dx = \int_{-L/2}^{L/2} \varepsilon_x dx$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Center-Loaded Beam

- Potential energy has three terms:
  - Bending strain energy
  - Stretching strain energy
  - External work

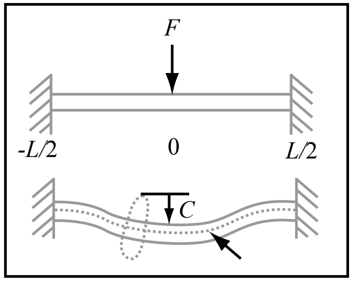


Image by MIT OpenCourseWare.

Adapted from Figure 10.2 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 250. ISBN: 9780792372462.

- Bending and external work already calculated for one trial function
- Pick another trial function (same weakness as last attempt, but easy to use) and include large deflections

$$\hat{w} = \frac{c}{2}\left(1 + \cos\frac{2\pi x}{L}\right)$$

Why not a û?

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Center-Loaded Beam

- First, calculate the strain due to stretching (aggregate axial strain)

$$\varepsilon_a = \frac{\delta L}{L} = \frac{1}{L}\int_{-L/2}^{L/2} \varepsilon_x dx$$

$$\varepsilon_a = \frac{1}{L}\int_{-L/2}^{L/2} \left[\frac{d\hat{u}}{dx} + \frac{1}{2}\left(\frac{d\hat{w}}{dx}\right)^2 \right] dx$$

$$\varepsilon_a = \frac{1}{L}\cancel{\left[\hat{u}\left(\frac{L}{2}\right) - \hat{u}\left(-\frac{L}{2}\right) \right]} + \frac{1}{L}\int_{-L/2}^{L/2} \left[\frac{1}{2}\left(\frac{d\hat{w}}{dx}\right)^2 \right] dx$$

- Total strain = bending strain + aggregate axial strain

$$\varepsilon_T = \varepsilon_{bending} + \varepsilon_{stretching}$$

$$\varepsilon_T = -z\frac{d^2\hat{w}}{dx^2} + \varepsilon_a$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Center-Loaded Beam

- Calculate total stored elastic energy from total strain

$$\mathrm{W} = \frac{EW}{2}\int_{-H/2}^{H/2}\int_{-L/2}^{L/2}\varepsilon_T^2\,dxdz = \frac{EWH\pi^4\left(8H^2+3c^2\right)c^2}{96L^3}$$

- Finally, potential energy…

$$U = \mathrm{W} - Fc = \frac{EWH\pi^4\left(8H^2+3c^2\right)c^2}{96L^3} - Fc$$

- …which we minimize with respect to c

$$\frac{\partial U}{\partial c} = 0 \quad \Rightarrow \quad F = \left(\frac{\pi^4}{6}\right)\left[\frac{EWH^3}{L^3}\right]c + \left(\frac{\pi^4}{8}\right)\left[\frac{EWH}{L^3}\right]c^3$$

- Compare linear term with solution to beam equation: prefactor 16.2 instead of 16

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Results from example

- Force-displacement relationship: an amplitude-stiffened Duffing spring

$$F = \left(\frac{\pi^4}{6}\right)\left[\frac{EWH^3}{L^3}\right]c + \left(\frac{\pi^4}{8}\right)\left[\frac{EWH}{L^3}\right]c^3$$

- Solution shows geometry dependence; constants may or may not be correct

$$F = C_b\left[\frac{EWH^3}{L^3}\right]c + C_s\left[\frac{EWH}{L^3}\right]c^3$$

- Once you’ve found the elastic strain energy, finding results for another load is easy

$$Work = Fc \quad \Longrightarrow \quad Work = \int_{-L/2}^{L/2} q(x)\hat{w}(x)dx$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: uniform pressure load P

- Adopt the elastic strain energy
- Calculate the work for a uniform pressure load

$$Work = WP \int_{-L/2}^{L/2} \frac{c}{2}\left(1+\cos\frac{2\pi x}{L}\right)dx = \frac{WLPc}{2}$$

- Minimize U to find relationship between load and deflection

$$P = \left(\frac{\pi^4}{3}\right)\left[\frac{EH^3}{L^4}\right]c + \left(\frac{\pi^4}{4}\right)\left[\frac{EH}{L^4}\right]c^3$$

- The geometry dependence appears!

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Combining Variational and FEM Methods

- Use the analytic variational method to find a good functional form for the result
- Establish non-dimensional numerical parameters within the solution
- Perform well-meshed FEM simulations over the design space
- Fit the analytic solution to the FEM results

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Residual Stress In Clamped Structures

- Must add a new term to the elastic energy to capture the effects of the residual stress

$$\widetilde{W} = \int_0^{\varepsilon_a} \sigma d\varepsilon \quad \Rightarrow \quad \widetilde{W} = \int_0^{\varepsilon_a} (\sigma_0 + E\varepsilon) d\varepsilon$$

- Now there is a residual stress term in the stored elastic energy

$$W_r = \sigma_0 W \int_{-H/2}^{H/2} dz \int_{-L/2}^{-L/2} \varepsilon_a dx$$

- For the fixed-fixed beam example, the residual stress term is:

$$W_r = \sigma_0 WLH \left(\frac{\pi^2}{4L^2} \right) c^2$$

- This leads to a general form of the load-deflection relationship for beams, which can be extended to plates and membranes

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Results for Doubly-Clamped Beam

For the case of a central point load :

$$F = \left\{ \left(\frac{\pi^2}{2} \right) \left[\frac{\sigma_0 WH}{L} \right] + \left(\frac{\pi^4}{6} \right) \left[\frac{EWH^3}{L^3} \right] \right\} c + \left(\frac{\pi^4}{8} \right) \left[\frac{EWH}{L^3} \right] c^3$$

and for the pressure loaded case :

$$P = \left\{ \pi^2 \left[\frac{\sigma_0 H}{L^2} \right] + \left(\frac{\pi^4}{3} \right) \left[\frac{EH^3}{L^4} \right] \right\} c + \left(\frac{\pi^4}{4} \right) \left[\frac{EH}{L^4} \right] c^3$$

The general form for pressure loading, useful for fitting to FEM results, is :

$$P = \left\{ C_r \left[\frac{\sigma_0 H}{L^2} \right] + C_b \left[\frac{EH^3}{L^4} \right] \right\} c + C_s \left[\frac{EH}{L^4} \right] c^3$$

Finally, we note that the stress term dominates over bending when

$$\sigma_0 \geq \frac{EH^2}{L^2}$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Outline

> Effects of residual stresses on structures

> Energy methods

- Elastic energy
- Principle of virtual work: variational methods
- Examples

> Rayleigh-Ritz methods for resonant frequencies and extracting lumped-element masses for structures

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Estimating Resonance Frequencies

- We have achieved part of our goal of converting structures into lumped elements
  - We can calculate elastic stiffness of almost any structure, for small and large deflections
  - But we still don't know how to find the mass term associated with structures
- We can get the mass term from the resonance frequency and the stiffness
- The resonance frequency comes from Rayleigh-Ritz analysis
  - In simple harmonic motion at resonance, the maximum kinetic energy equals the maximum potential energy
  - Determine kinetic energy; equate its maximum value to the maximum potential energy; find ω_0.

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Estimating Resonance Frequencies

- Guess a time dependent trial function from ŵ(x)

$$\hat{w}(x,t) = \hat{w}(x)\cos(\omega t)$$

- Find maximum kinetic energy from maximum velocity

$$\text{Maximum velocity}: \left(\frac{\partial \hat{w}(x,t)}{\partial t}\right)_{t=\pi/2\omega} = -\omega\hat{w}(x)$$

$$\text{Max kinetic energy, lumped}: \mathrm{W}_{k,\max} = \frac{1}{2} m v_{\max}^{2}$$

$$\text{Max kinetic energy density}: \widetilde{\mathrm{W}}_{k,\max} = \frac{1}{2}\rho_m(x)\omega^2\hat{w}(x)^2$$

$$\text{Max kinetic energy}: \mathrm{W}_{k,\max} = \frac{\omega^2}{2}\iiint\limits_{\text{volume beam}}\rho_m(x)\hat{w}(x)^2\,dxdydz$$

- Calculate maximum potential energy from ŵ(x) as before

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Rayleigh-Ritz

- The resonance frequency is obtained from the ratio of potential energy to kinetic energy, using a variational trial function
- The result is remarkably insensitive to the specific trial function

$$\omega_0^2 = \frac{\mathrm{W}_{elastic}}{\frac{1}{2}\iiint\limits_{Volume} \rho_m(x)\hat{w}^2(x)dx}$$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Example: Tensioned Beam

> Compare two trial solutions:

- Tensioned wire – the exact solution (1/2 λ of a cosine)
- Bent beam – a very poor solution

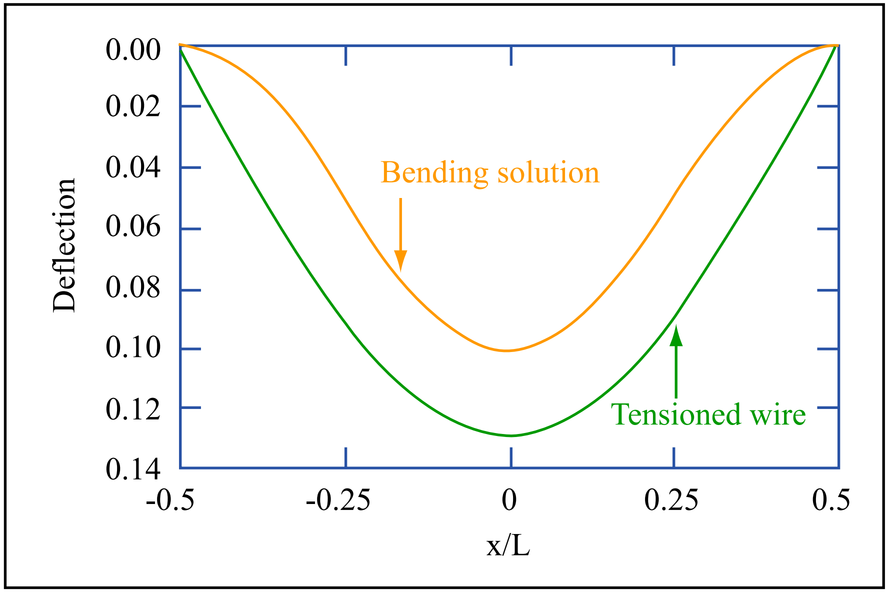


Image by MIT OpenCourseWare.
Adapted from Figure 10.3 in Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 263. ISBN: 9780792372462.

> Resonant frequencies differ by only 15%

> Worse trial functions yield higher stiffness, higher resonant frequencies

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].

Extracting Lumped Masses

- Use variational methods to calculate the stiffness
- Use Rayleigh-Ritz with the same trial function to calculate the resonant frequency ω^2
- Extract the mass from the relation between mass, stiffness, and resonant frequency.
  - $\omega^2 = k/m$

Cite as: Carol Livermore, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology. Downloaded on [DD Month YYYY].