Structures

Carol Livermore

Massachusetts Institute of Technology

* **With thanks to Steve Senturia, from whose lecture notes some of these materials are adapted.**

Outline

> **Regroup**

> **Beam bending**

- **Loading and supports**
- **Bending moments and shear forces**
- **Curvature and the beam equation**
- **Examples: cantilevers and doubly supported beams**

> **A quick look at torsion and plates**

Recall: Isotropic Elasticity

- For a general case of loading, the constitutive relationships between stress and elastic strain are as follows
- 6 equations, one for each normal stress and shear stress

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx}$$

Shear modulus G is given by $G = \frac{E}{2(1+\nu)}$

What we are considering today

- Bending in the limit of small deflections
- For axial loading, deflections are small until something bad happens
  - Nonlinearity, plastic deformation, cracking, buckling
  - Strains typically of order 0.1% to 1%
- For bending, small deflections are typically less than the thickness of the element (i.e. beam, plate) in question

What we are NOT considering today

> Basically, anything that makes today's theory not apply (not as well, or not at all)

> Large deflections
- Axial stretching becomes a noticeable effect

> Residual stresses
- Can increase or decrease the ease of bending

Our trajectory

- What are the loads and the supports?

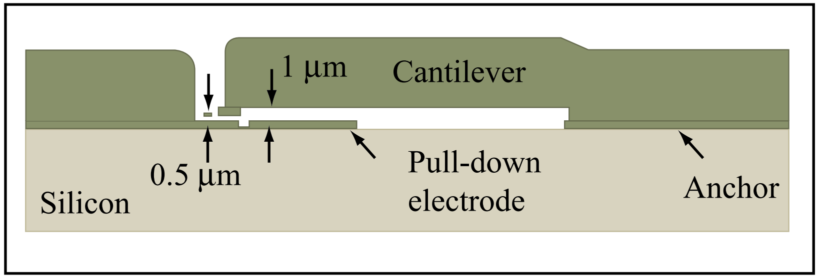
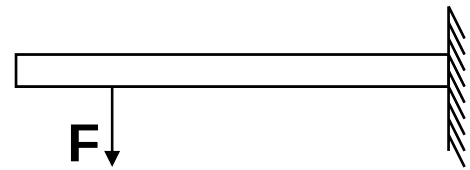
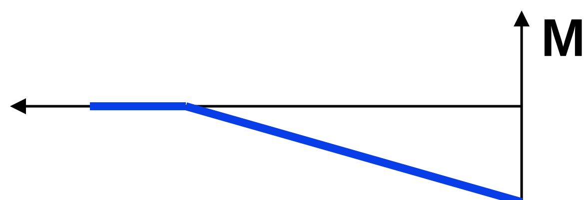


Image by MIT OpenCourseWare.

Adapted from Rebeiz, Gabriel M. *RF MEMS: Theory, Design, and Technology*. Hoboken, NJ: John Wiley, 2003. ISBN: 9780471201694.



- What is the bending moment at point x along the beam?



- How much curvature does that bending moment create in the structure at x? (Now you have the beam equation.)

- Integrate to find deformed shape

Outline

> **Regroup**

> **Bending**

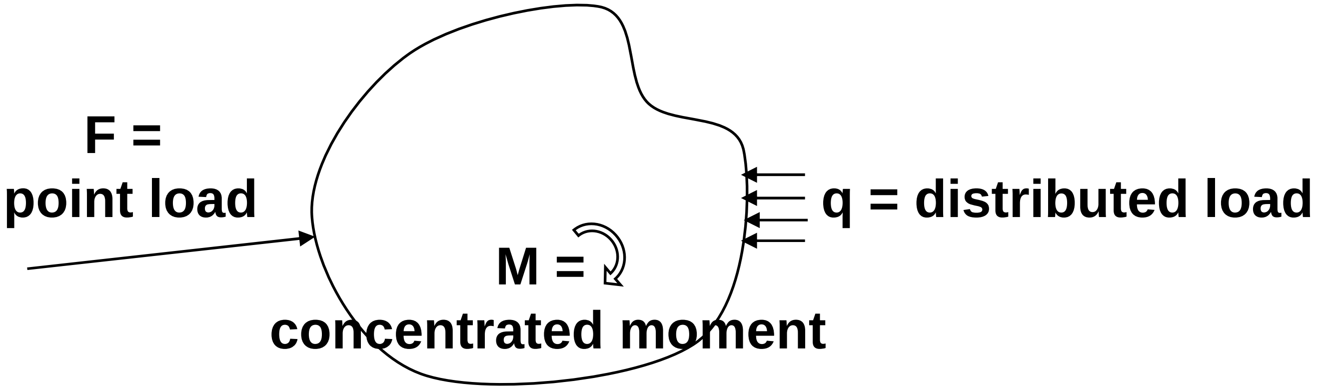
- **Loading and supports**
- **Bending moments and shear forces**
- **Curvature and the beam equation**
- **Examples: cantilevers and doubly supported beams**

> **A quick look at torsion and plates**

Types of Loads

> **Three basic types of loads:**

- **Point force (an old friend, with its own specific point of application)**
- **Distributed loads (pressure)**
- **Concentrated moment (what you get from a screwdriver, with a specific point of application)**
- **The forces and moments work together to make internal bending moments – more on this shortly**



Types of supports

> **Four basic boundary conditions:**

- **Fixed: can't translate at all, can't rotate**
- **Pinned: can't translate at all, but free to rotate (like a hinge)**
- **Pinned on rollers: can translate along the surface but not off the surface, free to rotate**
- **Free: unconstrained boundary condition**

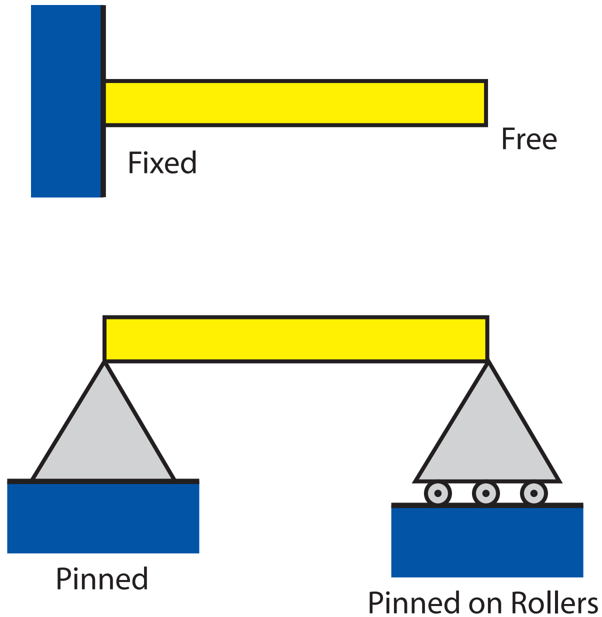


Image by MIT OpenCourseWare.
Adapted from Figure 9.5 in: Senturia, Stephen D. *Microsystem Design*.
Boston, MA: Kluwer Academic Publishers, 2001, p. 207. ISBN: 9780792372462.

Reaction Forces and Moments

> **Equilibrium requires that the total force on an object be zero and that the total moment about any axis be zero**

> **This gives rise to reaction forces and moments**

> **"Can't translate" means support can have reaction forces**

> **"Can't rotate" means support can have reaction moments**

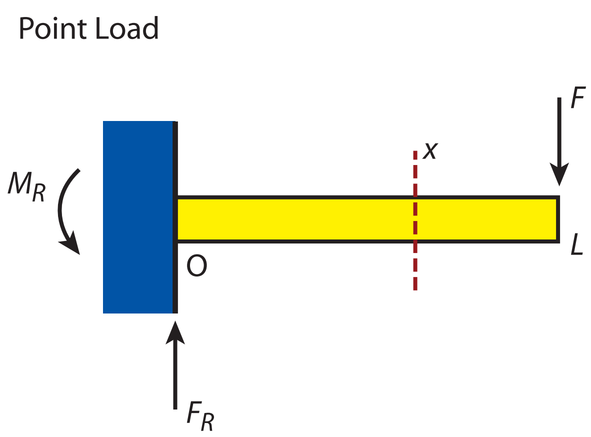


Image by MIT OpenCourseWare.
Adapted from Figure 9.7 in: Senturia, Stephen D. *Microsystem Design*.
Boston, MA: Kluwer Academic Publishers, 2001, p. 209. ISBN: 9780792372462.

Total moment about support:

$$M_T = M_R - FL$$

Moment must be zero in equilibrium:

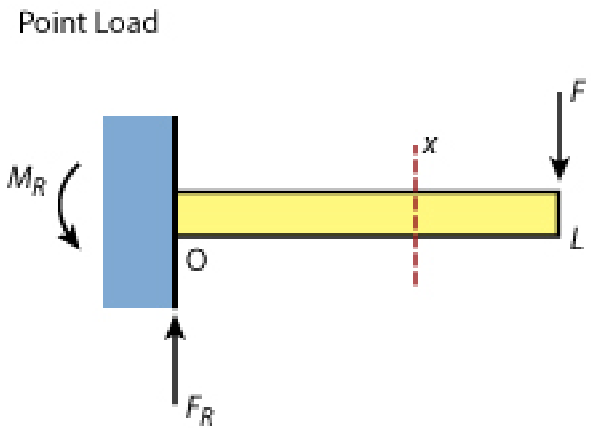
$$M_R = FL$$

Net force must be zero in equilibrium:

$$F_R = F$$

Internal forces and moments

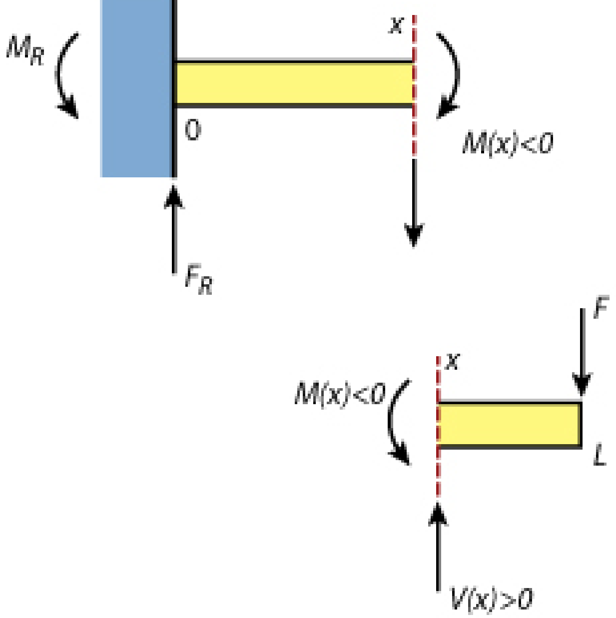
- Each segment of beam must also be in equilibrium
- This leads to internal shear forces V(x) and bending moments M(x)



For this case,

$$M(x) = -F(L-x)$$

$$V(x) = F$$



Images by MIT OpenCourseWare.

Adapted from Figure 9.7 in Senturia, Stephen D. *Microsystem Design*
Boston, MA: Kluwer Academic Publishers, 2001, p. 209. ISBN: 9780792372462.

Some conventions

Moments:

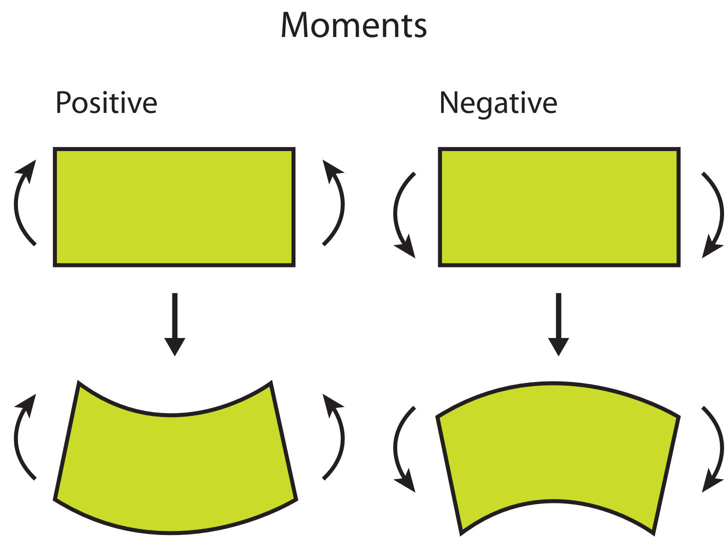


Image by MIT OpenCourseWare.
Adapted from Figure 9.8 in: Senturia, Stephen D.
Microsystem Design. Boston, MA: Kluwer Academic
Publishers, 2001, p. 210. ISBN: 9780792372462.

Shears:

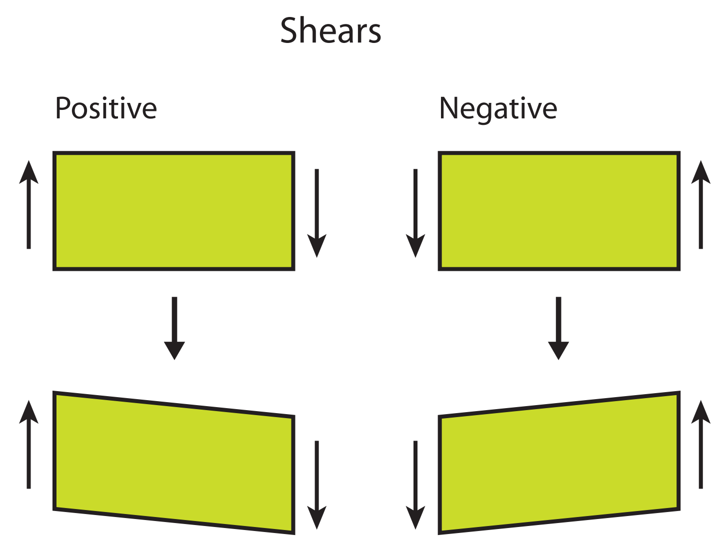


Image by MIT OpenCourseWare.
Adapted from Figure 9.8 in: Senturia, Stephen D.
Microsystem Design. Boston, MA: Kluwer Academic
Publishers, 2001, p. 210. ISBN: 9780792372462.

Combining all loads

> **A differential beam element, subjected to point loads, distributed loads and moments in equilibrium, must obey governing differential equations**

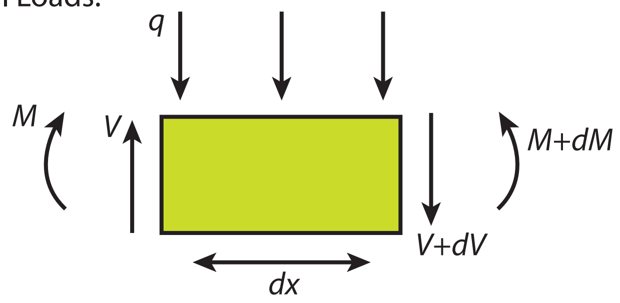


Image by MIT OpenCourseWare.
Adapted from Figure 9.8 in: Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, p. 210. ISBN: 9780792372462.

$$F_T = qdx + (V + dV) - V$$

$$\Rightarrow \frac{dV}{dx} = -q$$

$$M_T = (M + dM) - M - (V + dV)dx - \cancel{\frac{qdx}{2}}dx$$

$$\Rightarrow \frac{dM}{dx} = V$$

Outline

> **Regroup**

> **Bending**

- **Loading and supports**
- **Bending moments and shear forces**
- **Curvature and the beam equation**
- **Examples: cantilevers and doubly supported beams**

> **A quick look at torsion and plates**

Pure bending

> **Important concept: THE NEUTRAL AXIS**

> **Axial stress varies with transverse position relative to the neutral axis**

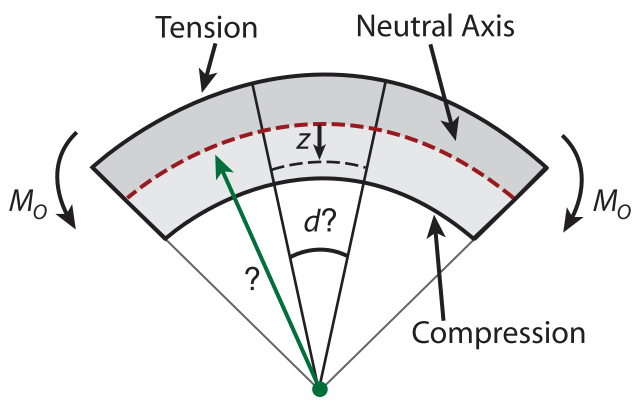
$$dL = (\rho - z)d\theta$$

$$dx = \rho d\theta$$

$$\Rightarrow dL = dx - \frac{z}{\rho}dx$$

$$\varepsilon_x = -\frac{z}{\rho}$$

$$\sigma_x = -\frac{zE}{\rho}$$



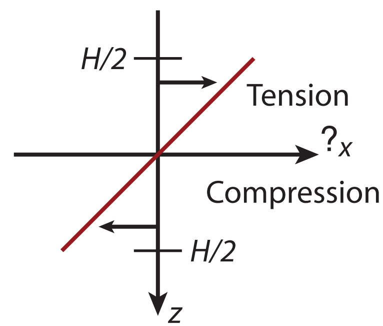


Image by MIT OpenCourseWare.
Adapted from Figures 9.9 and 9.10 in: Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, pp. 211, 213. ISBN: 9780792372462.

Locating the neutral axis

- In pure bending, locate the neutral axis by imposing equilibrium of axial forces

$$N = \int_A \sigma(z)\,dA = 0$$

$$-\int_{thickness} \frac{E(z)W(z)z}{\rho}dz = 0$$

One material, rectangular beam

$$-\frac{EW}{\rho}\int_{thickness} z\,dz = 0$$

- The neutral axis is in the middle for a one material beam of symmetric cross-section.
- Composite beams: if the beam just has a very thin film on it, can approximate neutral axis unchanged
- Composite beams: with films of comparable thickness, change in E biases the location of the neutral axis

Curvature in pure bending

Curvature is related to the internal bending moment M.

Internal Moment:

$$M = \int_A z\sigma_x dA$$

$$\sigma_x = -\frac{zE}{\rho}$$

$$M = -\frac{1}{\rho}\int_A E(z)z^2 dA$$

For a one-material beam,

$$M = -\frac{E}{\rho}\int_A z^2 dA$$

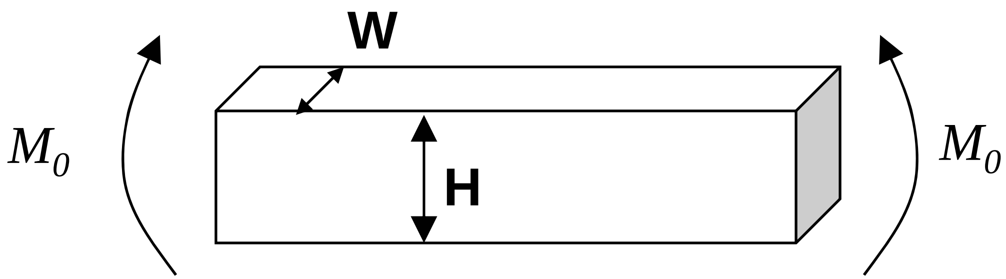
Moment of inertia I:

$$I = \int_A z^2 dA$$

$$M = -\frac{EI}{\rho}$$

In pure bending, the internal moment M equals the externally applied moment M_0. Then $\frac{1}{\rho} = -\frac{M_0}{EI}$ **for one material; for two or more materials, calculate an effective EI.**

Useful case: rectangular beam



For a uniform rectangular beam,

$$I = \int_{-H/2}^{H/2} W z^2 dz = \frac{1}{12} W H^3$$

$$M = -\left(\frac{1}{12} W H^3 \right) \frac{E}{\rho}$$

Differential equation of beam bending

> Relation between curvature and the applied load

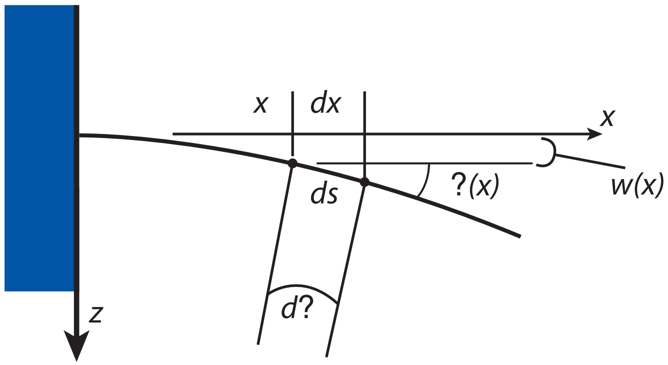


Image by MIT OpenCourseWare.

Adapted from Figure 9.11 in: Senturia, Stephen D. *Microsystem Design*. Boston, MA: Kluwer Academic Publishers, 2001, pp. 214. ISBN: 9780792372462.

$$ds = \frac{dx}{\cos\theta} \approx dx$$

$$\frac{dw}{dx} = \tan\theta \approx \theta$$

$$ds = \rho d\theta$$

$$\Downarrow$$

$$\frac{d\theta}{dx} \approx \frac{1}{\rho} = \frac{d^2 w}{dx^2}$$

$$\boxed{\frac{d^2 w}{dx^2} = -\frac{M}{EI}}$$

and, by successive differentiation

$$\frac{d^3 w}{dx^3} = -\frac{V}{EI}$$

$$\boxed{\frac{d^4 w}{dx^4} = \frac{q}{EI}}$$

For large - angle bending

$$\frac{1}{\rho} = \frac{w''}{\left[1 + (w')^2\right]^{3/2}}$$

Large-angle bending is rare in MEMS structures

Anticlastic curvature

- If a beam is bent, then the Poisson effect causes opposite bending in the transverse direction

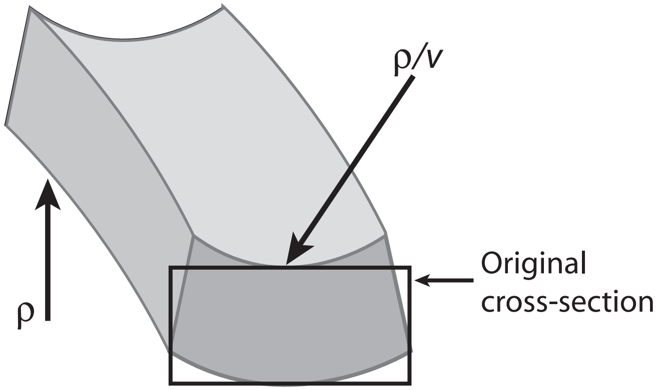


Image by MIT OpenCourseWare.
Adapted from Figure 9.12 in: Senturia, Stephen D. *Microsystem Design*.
Boston, MA: Kluwer Academic Publishers, 2001, pp. 219. ISBN: 9780792372462.

$$\varepsilon_x = -\frac{z}{\rho}$$

$$\varepsilon_y = -\nu\varepsilon_x$$

$$\Downarrow$$

$$\varepsilon_y = \frac{\nu z}{\rho}$$

which creates a y - directed internal moment

Example: Cantilever with point load

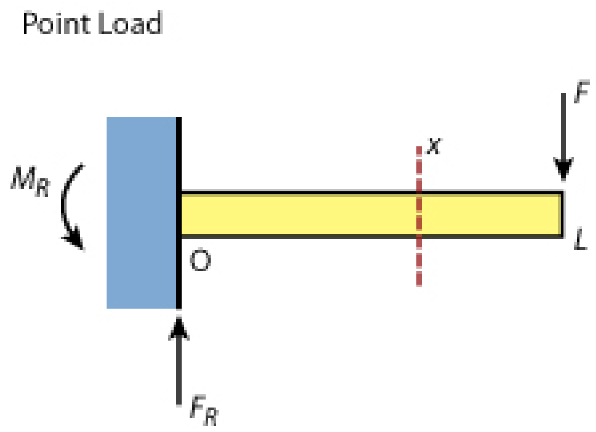


Image by MIT OpenCourseWare.
Adapted from Figure 9.7 in: Senturia, Stephen D. *Microsystem Design*.
Boston, MA: Kluwer Academic Publishers, 2001, p. 209. ISBN: 9780792372462.

$$\frac{d^2w}{dx^2} = -\frac{M(x)}{EI}$$

and

$$M(x) = -F(L-x)$$

$$V(x) = F$$

$$\frac{d^2w}{dx^2} = \frac{F}{EI}(L-x)$$

$$\frac{dw}{dx} = -\frac{F}{2EI}x^2 + \frac{FL}{EI}x + A$$

$$w = -\frac{F}{6EI}x^3 + \frac{FL}{2EI}x^2 + Ax + B$$

$$\text{BC}: \quad w(0) = 0, \ \left.\frac{dw}{dx}\right|_{x=0} = 0$$

$$A = B = 0$$

$$w = \frac{FL}{2EI}x^2\left(1 - \frac{x}{3L}\right)$$

Spring Constant for Cantilever

- **Since force is applied at tip, if we find maximum tip displacement, the ratio of displacement to force is the spring constant.**

$$w_{\max} = \left(\frac{L^3}{3EI}\right)F$$

$$\Downarrow$$

$$k_{cantilever} = \frac{3EI}{L^3} = \frac{EWH^3}{4L^3}$$

For the same dimensions as the uniaxially loaded beam,

$$k_{cantilever} = 0.2\ \mathrm{N/m}$$

Stress in the Bent Cantilever

> **To find bending stress, we find the radius of curvature, then use the pure-bending case to find stress**

$$\text{Radius of curvature}: \ \frac{1}{\rho} = \frac{d^2 w}{dx^2} = \frac{F}{EI}(L - x)$$

$$\text{Maximum value is at support } (x = 0)$$

$$\left.\frac{1}{\rho}\right|_{\max} = \frac{FL}{EI}$$

$$\text{Maximum axial strain is at surface } \left(\mathrm{z} = \mathrm{H}/2\right)$$

$$\varepsilon_{\max} = \frac{LH}{2EI} F = \frac{6L}{H^2 WE} F$$

$$\sigma_{\max =} \frac{6L}{H^2 W} F$$

Tabulated solutions

> **Solutions to simple situations available in introductory mechanics books**
  - **Point loads, distributed loads, applied moments**
  - **Handout from Crandall, Dahl, and Lardner, An Introduction to the Mechanics of Solids, 1999, p. 531.**

> **Linearity: you can superpose the solutions**

> **Can save a bit of time**

> **Solutions use nomenclature of singularity functions**

Singularity functions

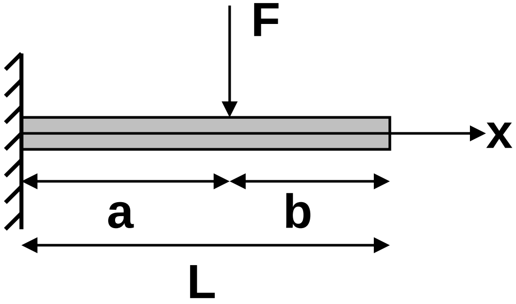
$$\langle x-a \rangle^n = 0 \text{ if } x-a<0$$

$$\langle x-a \rangle^n = (x-a)^n \text{ if } x-a>0$$

$$\langle x-a \rangle^{-1} = \text{what is variably called an impulse or a delta function}$$

Integrate as if it were just functions of (x-a); evaluate at the end.

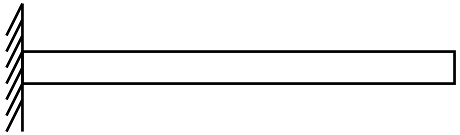
Value: a single expression describes what's going on in different regions of the beam



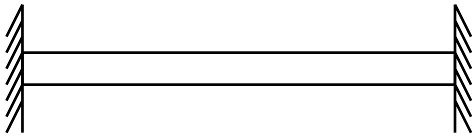
$$M(x) = -F\langle a-x \rangle$$

Overconstraint

> A cantilever's single support provides the necessary support reactions and no more



> A fixed-fixed beam has an additional support, so it is overconstrained



> Static indeterminacy: must consider deformations and reactions to determine state of the structure

> Many MEMS structures are statically indeterminate: flexures, optical MEMS, switches,…

> What this means for us

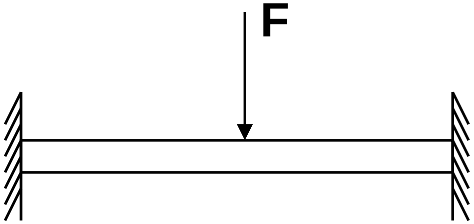
- Failure modes and important operational effects: stress stiffening, buckling
- Your choice of how to calculate deflections

Example: center-loaded fixed-fixed beam

> **Option 1: (general)**

- **Start with beam equation in terms of q**
- **Express load as a delta function**
- **Integrate four times**
- **Four B.C. give four constants**

$$\frac{d^4 w}{dx^4} = \frac{q}{EI}$$



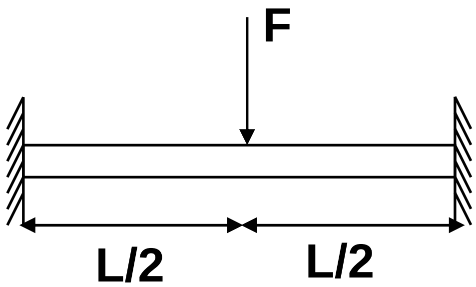
> **Option 2: (not general)**

- **Invoke symmetry**

> **Option 3: (general)**

- **Pretend beam is a cantilever with as yet unknown moment and force applied at end such that w(L) = slope(L) = 0**
- **Using superposition, solve for deflection and slope everywhere**
- **Impose B.C. to determine moment and force at end**
- **Plug newly-determined moment and force into solution, and you're done**

Integration using singularity functions



$$q = F\left\langle x - L/2 \right\rangle^{-1}$$

$$\frac{d^4 w}{dx^4} = \frac{q}{EI} = \frac{F}{EI}\left\langle x - L/2 \right\rangle^{-1}$$

$$\frac{d^3 w}{dx^3} = \frac{F}{EI}\left\langle x - L/2 \right\rangle^{0} + C_1$$

$$\frac{d^2 w}{dx^2} = \frac{F}{EI}\left\langle x - L/2 \right\rangle^{1} + C_1 x + C_2$$

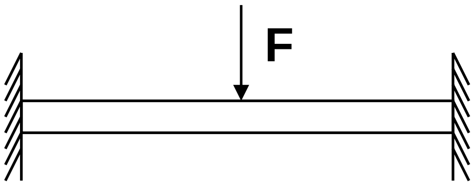
$$\frac{dw}{dx} = \frac{F}{EI}\frac{\left\langle x - L/2 \right\rangle^{2}}{2} + \frac{C_1 x^2}{2} + C_2 x + C_3$$

$$w(x) = \frac{F}{EI}\frac{\left\langle x - L/2 \right\rangle^{3}}{6} + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4$$

Use boundary conditions to find constants: no displacement at supports, slope = 0 at supports

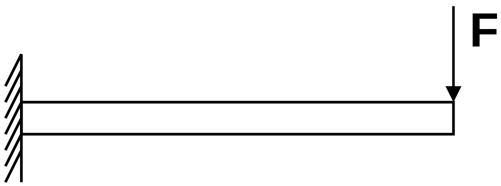
Comparing spring constants

- Center-loaded fixed-fixed beam (same dimensions as previous)



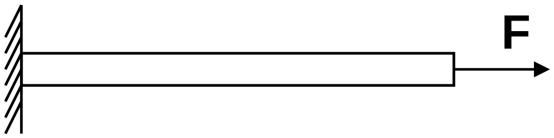
$$k = \frac{16EWH^3}{L^3} = 12.8 \text{ N/m}$$

- Tip-loaded cantilever beam, same dimensions



$$k = \frac{EWH^3}{4L^3} = 0.2 \text{ N/m}$$

- Uniaxially loaded beam, same dimensions



$$k = \frac{EWH}{L} = 8000 \text{ N/m}$$

Outline

> **Regroup**

> **Bending**

- **Loading and supports**
- **Bending moments and shear forces**
- **Curvature and the beam equation**
- **Examples: cantilevers and doubly supported beams**

> **A quick look at torsion and plates**

Torsion

The treatment of torsion mirrors that of bending.

Figures 48 and 50 in: Hornbeck, Larry J. "From Cathode Rays to Digital Micromirrors: A History of Electronic Projection Display Technology." *Texas Instruments Technical Journal* 15, no. 3 (July-September 1998): 7-46.

Figure 51 on p. 39 in: Hornbeck, Larry J. "From Cathode Rays to Digital Micromirrors: A History of Electronic Projection Display Technology." *Texas Instruments Technical Journal* 15, no. 3 (July-September 1998): 7-46.

Digital Light Processing™ Technology: Texas Instruments

Bending of plates

- A plate is a beam that is so wide that the transverse strains are inhibited, both the Poisson contraction and its associated anticlastic curvature
- This leads to additional stiffness when trying to bend a plate

$$\varepsilon_x = \frac{\sigma_x - \nu\sigma_y}{E}$$

But ε_y is constrained to be zero

$$\Rightarrow 0 = \varepsilon_y = \frac{\sigma_y - \nu\sigma_x}{E}$$

$$\Downarrow$$

$$\sigma_x = \left(\frac{E}{1-\nu^2}\right)\varepsilon_x$$

Plate Modulus

Plate in pure bending

- Analogous to beam bending, with the limit on transverse strains
- Two radii of curvature along principal axes

$$\frac{1}{\rho_x} = \frac{\partial^2 w}{\partial x^2} \qquad \varepsilon_x = -\frac{z}{\rho_x} \qquad \varepsilon_x = \frac{1}{E}\left(\sigma_x - \upsilon\sigma_y\right)$$

$$\frac{1}{\rho_y} = \frac{\partial^2 w}{\partial y^2} \qquad \varepsilon_y = -\frac{z}{\rho_y} \qquad \varepsilon_y = \frac{1}{E}\left(\sigma_y - \upsilon\sigma_x\right)$$

- Stresses along principal axes

$$\sigma_x = -\frac{Ez}{\left(1-\nu^2\right)}\left(\frac{1}{\rho_x} + \frac{\nu}{\rho_y}\right)$$

$$\sigma_y = -\frac{Ez}{\left(1-\nu^2\right)}\left(\frac{1}{\rho_y} + \frac{\nu}{\rho_x}\right)$$

Plate in pure bending

> **Relate moment per unit width of plate to curvature**

> **Treat x and y equivalently**

$$\frac{M_x}{W} = \int_{thickness} z\sigma_x(z)dz$$

$$\frac{M_x}{W} = -\frac{E}{(1-\nu^2)}\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right)\int_{-H/2}^{H/2} z^2 dz$$

$$\frac{M_x}{W} = -D\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right)$$

$$\text{where} \quad D = \frac{1}{12}\left(\frac{EH^3}{1-\nu^2}\right)$$

flexural rigidity

> **Note that stiffness comes from flexural rigidity as for a beam**

Plate in pure bending

- Recall that M is two derivatives away from a distributed load, and that

$$\frac{1}{\rho_x} = \frac{\partial^2 w}{\partial x^2} \qquad \frac{1}{\rho_y} = \frac{\partial^2 w}{\partial y^2}$$

- This leads to the equation for small amplitude bending of a plate

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = P(x, y)$$

distributed load

- Often solve with polynomial solutions (simple cases) or eigenfunction expansions

Where are we now?

- We can handle small deflections of beams and plates
- Physics intervenes for large deflections and residual stress, and our solutions are no longer correct
- Now what do we do?
  - Residual stress: include it as an effective load
  - Large deflections: use Energy Methods