Generalizations: Adding an extra parameter

In the discussion of implementing the Fourier Transform and the Inverse Fourier Transform, $\Psi(x)$ and $A(q)$, were always described as vectors, indexed by **x** and **q** respectively. In general, these each could have been described by a matrix, again, with a row index of **x** and **q**, but with a column index of some independent parameter, like time.

For example, Ψ could have been a matrix. Each column could be for a different time instance:

$$\Psi = \begin{pmatrix} \Psi(x_1,t_1) & \Psi(x_1,t_2) & \cdots & \Psi(x_1,t_l) \\ \Psi(x_2,t_1) & \Psi(x_2,t_2) & \cdots & \Psi(x_2,t_l) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(x_n,t_1) & \Psi(x_n,t_2) & \cdots & \Psi(x_n,t_l) \end{pmatrix}$$

then, applying the Fourier Transform procedures exactly as described previously, we would get a matrix $A(q,t)$ instead of the vector $\bar{A}(q)$:

$$A(q,t) = \begin{pmatrix} A(q_1,t_1) & A(q_1,t_2) & \cdots & A(q_1,t_l) \\ A(q_2,t_1) & A(q_2,t_2) & \cdots & A(q_2,t_l) \\ \vdots & \vdots & \ddots & \vdots \\ A(q_m,t_1) & A(q_m,t_2) & \cdots & A(q_m,t_l) \end{pmatrix}$$

The most probable scenario is that we are given the initial wavepacket $\Psi(x,t=0)$ and wish to find the wave packet at time **t>0**. In this case we perform the Fourier Transform on a single column, and are returned a single column amplitude function. Then, we want to find Ψ for many time instances. Using matrices instead of vectors, we can compute all the time instances at once. First, we setup a matrix A:

$$A = \begin{pmatrix} A(q_1) \\ A(q_2) \\ \vdots \\ A(q_m) \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & \cdots & 1_m \end{pmatrix}$$

$$= \begin{pmatrix} A(q_1) & A(q_1) & \cdots & A(q_1) \\ A(q_2) & A(q_2) & \cdots & A(q_2) \\ \vdots & \vdots & \ddots & \vdots \\ A(q_m) & A(q_m) & \cdots & A(q_m) \end{pmatrix}$$

This is a matrix, with identical column. Each column is the expansion coefficients we computed from the Fourier transform of Ψ . We will put one column in the matrix for each future time instance we wish to compute Ψ at.

If we then similarly redefine $\vec{\varsigma}$ to account for the time parameter as follows:

$$\vec{\varsigma} = \begin{pmatrix} e^{-iE_1t_1/\hbar} & e^{-iE_1t_2/\hbar} & \cdots & e^{-iE_1t_l/\hbar} \\ e^{-iE_2t_1/\hbar} & e^{-iE_2t_2/\hbar} & \cdots & e^{-iE_2t_l/\hbar} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-iE_mt_1/\hbar} & e^{-iE_mt_2/\hbar} & \cdots & e^{-iE_mt_l/\hbar} \end{pmatrix}$$

Now, if we perform array multiplication (element by element) on ς and A we get:

$$A(q,t) = \begin{pmatrix} A(q_1)e^{-it_1E_1/\hbar} & A(q_1)e^{-it_2E_1/\hbar} & A(q_1)e^{-it_3E_1/\hbar} & \cdots & A(q_1)e^{-it_lE_1/\hbar} \\ A(q_2)e^{-it_1E_2/\hbar} & A(q_2)e^{-it_2E_2/\hbar} & A(q_2)e^{-it_3E_2/\hbar} & \cdots & A(q_2)e^{-it_lE_2/\hbar} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A(q_m)e^{-it_1E_m/\hbar} & A(q_m)e^{-it_2E_m/\hbar} & A(q_m)e^{-it_3E_m/\hbar} & \cdots & A(q_m)e^{-it_lE_m/\hbar} \end{pmatrix}$$

and, if we multiply $\vec{\phi}_{\bar{q}} \cdot A(q,t)$, just like we did in the previous sections:

$$= \begin{pmatrix} \left(A(q_1,t_1)e^{iq_1x_1} + \cdots + A(q_m,t_1)e^{iq_mx_1}\right) & \cdots & \left(A(q_1,t_l)e^{iq_1x_1} + \cdots + A(q_m,t_l)e^{iq_mx_1}\right) \\ \left(A(q_1,t_1)e^{iq_1x_2} + \cdots + A(q_m,t_1)e^{iq_mx_2}\right) & \cdots & \left(A(q_1,t_l)e^{iq_1x_2} + \cdots + A(q_m,t_l)e^{iq_mx_2}\right) \\ \vdots & \vdots & \vdots \\ \left(A(q_1,t_1)e^{iq_1x_n} + \cdots + A(q_m,t_1)e^{iq_mx_n}\right) & \cdots & \left(A(q_1,t_l)e^{iq_1x_n} + \cdots + A(q_m,t_l)e^{iq_mx_n}\right) \end{pmatrix}$$

which is just

$$\Psi(x,t) = \begin{pmatrix} \Psi(x_1,t_1) & \Psi(x_1,t_2) & \cdots & \Psi(x_1,t_l) \\ \Psi(x_2,t_1) & \Psi(x_2,t_2) & \cdots & \Psi(x_2,t_l) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(x_n,t_1) & \Psi(x_n,t_2) & \cdots & \Psi(x_n,t_l) \end{pmatrix}$$