6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

Final - Solutions - Spring 2009

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Problem 1

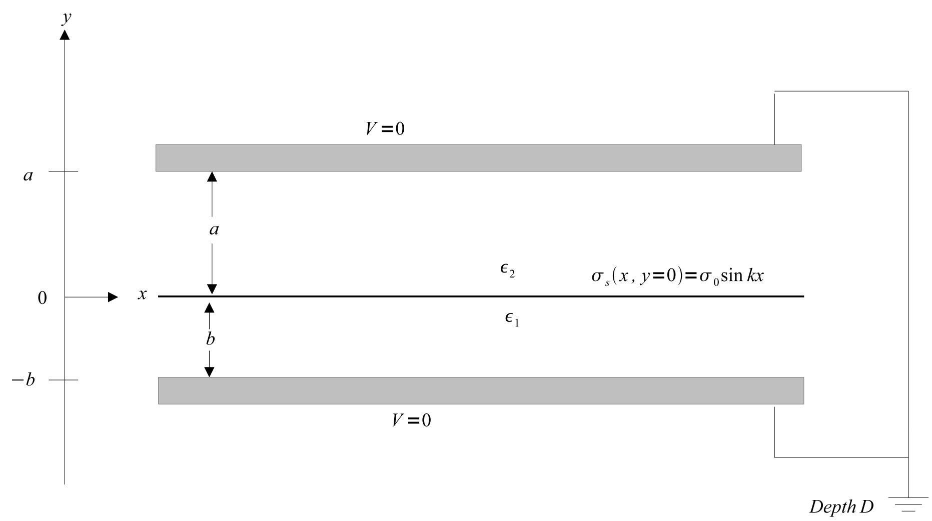

Figure 1: A diagram of a sheet of surface charge at $y = 0$ between two grounded perfect conductors at $y = -b$ and $y = a$ (Image by MIT OpenCourseWare).

A sheet of surface charge with surface charge distribution $\sigma_s(x, y = 0) = \sigma_0 \sin kx$ is placed at $y = 0$, parallel and between two parallel grounded perfect conductors at zero potential at $y = -b$ and $y = a$. The regions above and below the potential sheet have dielectric permittivities of ϵ_2 and ϵ_1. Neglect fringing field effects.

A

Question: What are the electric potential solutions in the regions $0 \leq y \leq a$ and $-b \leq y \leq 0$?
Solution:

$$\Phi(x, y) = \begin{cases} A \sinh k(y-a) \sin kx & 0 < y < a \\ B \sinh k(y+b) \sin kx & -b < y < 0 \end{cases}$$

$$\Phi(x, y = 0_-) = \Phi(x, y = 0_+) \Rightarrow -A \sinh ka = B \sinh kb$$

$$E_y(x, y = 0_+) = -\frac{\partial \Phi}{\partial y}\Big|_{y=0+} = -Ak \cosh k(y-a) \sin kx\Big|_{y=0_+} = -Ak \cosh ka \sin kx$$

$$E_y(x, y = 0_-) = -\frac{\partial \Phi}{\partial y}\Big|_{y=0-} = -Bk \cosh k(y+b) \sin kx\Big|_{y=0_-} = -Bk \cosh kb \sin kx$$

$$\sigma_s(x, y=0) = \sigma_0 \sin kx = \epsilon_2 E_y(x, y=0_+) - \epsilon_1 E_y(x, y=0_-) = -\epsilon_2 Ak \cosh ka \sin kx + \epsilon_1 Bk \cosh kb \sin kx$$

$$\sigma_0 = -\epsilon_2 Ak \cosh ka + \epsilon_1 Bk \cosh kb$$

$$B = -\frac{A \sinh ka}{\sinh kb} \Rightarrow -\epsilon_2 Ak \cosh ka - \epsilon_1 k \frac{A \sinh ka \cosh kb}{\sinh kb} = \sigma_0$$
$$\Rightarrow -Ak\big[\epsilon_2 \cosh ka + \frac{\epsilon_1 \sinh ka \cosh kb}{\sinh kb}\big] = \sigma_0$$

$$A = \frac{-\sigma_0 \sinh kb}{k[\epsilon_2 \cosh ka \sinh kb + \epsilon_1 \sinh ka \cosh kb]}$$

$$B = \frac{-A \sinh ka}{\sinh kb} = \frac{\sigma_0 \sinh ka}{k[\epsilon_2 \cosh ka \sinh kb + \epsilon_1 \sinh ka \cosh kb]}$$

$$\Phi(x,y) = \begin{cases} A \sinh k(y-a) \sin kx = \dfrac{-\sigma_0 \sinh kb \sinh k(y-a) \sin kx}{k[\epsilon_2 \cosh ka \sinh kb + \epsilon_1 \sinh ka \cosh kb]} & 0 < y < a \\ B \sinh k(y+b) \sin kx = \dfrac{\sigma_0 \sinh ka \sinh k(y+b) \sin kx}{k[\epsilon_2 \cosh ka \sinh kb + \epsilon_1 \sinh ka \cosh kb]} & -b < y < 0 \end{cases}$$

B

Question: What are the electric field distributions in the regions $0 < y < a$ and $-b < y < 0$?
Solution:

$$0 < y < a \quad \overline{E} = -\nabla\Phi = -\big[\frac{\partial \Phi}{\partial x}\overline{i_x} + \frac{\partial \Phi}{\partial y}\overline{i_y}\big] = -Ak\left[\cosh kx \sinh k(y-a)\bar{i}_x + \sin kx \cosh k(y-a)\bar{i}_y\right]$$
$$-b < y < 0 \quad \overline{E} = -Bk[\cos kx \sinh k(y+b)\overline{i_x} + \sin kx \cosh k(y+b)\overline{i_y}]$$

C

Question: What are the free surface charge distributions at $y = -b$ and $y = a$?
Solution:

$$\begin{aligned}\sigma_s(x, y=-b) &= \epsilon_2 E_y(x, y=-b) \\ &= -\epsilon_2 Bk \sin kx \\ &= \frac{-\sigma_0 \epsilon_2 \sinh ka \sin kx}{[\epsilon_2 \cosh ka \sinh kb + \epsilon_1 \sinh ka \cosh kb]}\end{aligned}$$

$$\begin{aligned}\sigma_s(x, y=a) &= -\epsilon_1 E_y(x, y=a) \\ &= \epsilon_1 Ak \sin kx \\ &= \frac{-\epsilon_1 \sigma_0 \sinh kb \sin kx}{[\epsilon_2 \cosh ka \sinh kb + \epsilon_1 \sinh ka \cosh kb]}\end{aligned}$$

D

Question: What is the potential distribution at $y = 0$?
Solution:

$$\Phi(x, y = 0) = \frac{\sigma_0 \sinh ka \sinh kb \sin kx}{k[\epsilon_2 \cosh ka \sinh kb + \epsilon_1 \sinh ka \cosh kb]}$$

Problem 2

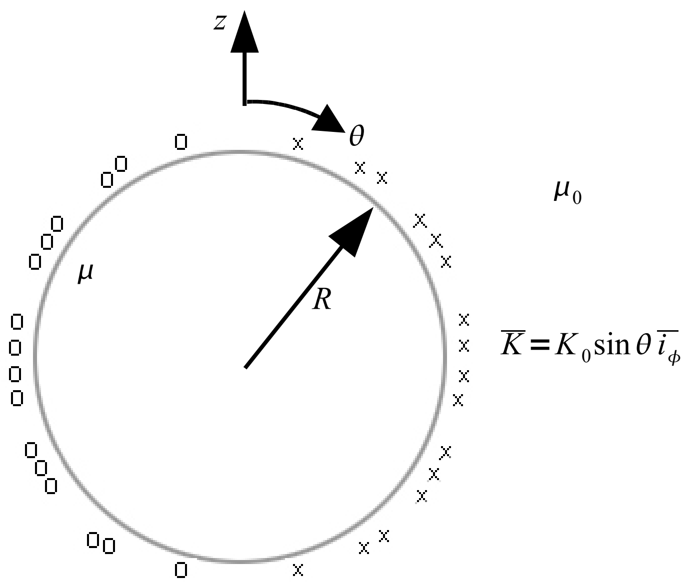

Figure 2: A diagram of a surface current sheet placed on the surface of a sphere of radius R (Image by MIT OpenCourseWare).

A surface current sheet $\overline{K} = K_0 \sin\theta \overline{i_\phi}$ is placed on the surface of a sphere of radius R. The inside of the sphere ($r < R$) has magnetic permeability μ and the outside region ($r > R$) is free space with magnetic permeability μ_0. The magnetic field at $r = \infty$ is zero.

A

Question: What are the boundary conditions on the magnetic field at $r = 0$ and $r = R$?
Solution:

$$\overline{H}(r = 0) = \text{ finite}, \qquad H_\theta(r = R_+, \theta) - H_\theta(r = R_-, \theta) = K_0 \sin\theta$$

$$\mu H_r(r = R_-, \theta) = \mu_0 H_r(r = R_+, \theta)$$

B

Question: What are the general form of the solutions for the magnetic scalar potential inside and outside the sphere?
Solution:

$$\nabla \times H = 0 \Rightarrow \overline{H} = -\nabla\chi$$

$$\chi(r,\theta) = \begin{cases} Ar\cos\theta & 0 < r < R \\ \frac{C}{r^2}\cos\theta & R < r < \infty \end{cases}$$

C

Question: Use the boundary conditions of part (a) and solve for the magnetic scalar potential and the magnetic field $\overline{H}$ inside and outside the sphere.
Solution:

$$H_\theta = -\frac{1}{r}\frac{\partial\chi}{\partial\theta} = \begin{cases} A\sin\theta & 0 < r < R \\ \frac{C}{r^3}\sin\theta & R < r < \infty \end{cases}$$

$$H_r = -\frac{\partial\chi}{\partial r} = \begin{cases} -A\cos\theta & 0 < r < R \\ \frac{2C}{r^3}\cos\theta & R < r < \infty \end{cases}$$

$$H_\theta(r = R_+,\theta) - H_\theta(r = R_-,\theta) = K_0\sin\theta \Rightarrow \frac{C}{R^3} - A = K_0$$

$$\mu H_r(r = R_-,\theta) - \mu_0 H_r(r = R_+,\theta) \Rightarrow -\mu A = \frac{\mu_0 2C}{R^3}$$

$$A = -\frac{\mu_0}{\mu}\frac{2C}{R^3} \Rightarrow \frac{C}{R^3}\left(1 + \frac{2\mu_0}{\mu}\right) = K_0$$

$$C = \frac{K_0 R^3}{1 + \frac{2\mu_0}{\mu}},\ \ A = -\frac{2\mu_0}{\mu}\frac{C}{R^3} = -\frac{2\mu_0}{\mu}\frac{K_0}{1 + \frac{2\mu_0}{\mu}}$$

$$\chi(r,\theta) = \begin{cases} -\frac{2\mu_0}{\mu}\frac{K_0}{1+\frac{2\mu_0}{\mu}} r\cos\theta & 0 < r < R \\ \frac{K_0 R^3}{1+\frac{2\mu_0}{\mu}}\frac{\cos\theta}{r^2} & R < r < \infty \end{cases}$$

$$\overline{H}(r,\theta) = \begin{cases} -A(\cos\theta\overline{i_r} - \sin\theta\overline{i_\theta}) = -A\overline{i_z} = \frac{2\mu_0}{\mu}\frac{K_0}{1+\frac{2\mu_0}{\mu}}\overline{i_z} & 0 < r < R \\ \frac{K_0}{1+\frac{2\mu_0}{\mu}}\frac{R^3}{r^3}(2\cos\theta\overline{i_r} + \sin\theta\overline{i_\theta}) & R < r < \infty \end{cases}$$

D

Question: The scalar magnetic potential for a point magnetic dipole of moment $m\overline{i_z}$ at the origin is $\overline{H} = -\nabla\chi$, $\chi = \frac{m\cos\theta}{4\pi r^2}$

What is the effective magnetic moment of the sphere and the surface current sheet for $r > R$?

Solution:

$$\frac{m}{4\pi} = \frac{K_0 R^3}{1 + \frac{2\mu_0}{\mu}} \Rightarrow m = \frac{4\pi K_0 R^3}{1 + \frac{2\mu_0}{\mu}}$$

E

Question: What is the equation for the magnetic field line that passes through the point $(r = R_0, \theta = \frac{\pi}{2})$ where $R_0 > R$.

Solution:

$$\frac{dr}{rd\theta} = \frac{H_r}{H_\theta} = \frac{2\cos\theta}{\sin\theta} \quad R < r < \infty$$

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta}$$

$$\int \frac{dr}{r} = \int \frac{2\cos\theta}{\sin\theta} d\theta$$

$$\ln r = \int \frac{2\cos\theta}{\sin\theta} d\theta$$

Let $u = \sin\theta, du = \cos\theta d\theta$

$$\ln r = \int \frac{2du}{u} = 2\ln u + C_1 = \ln u^2 + C_1 = \ln(\sin^2\theta) + C_1$$

$$\ln \frac{r}{\sin^2\theta} = C_1 \Rightarrow \frac{r}{\sin^2\theta} = e^{C_1} = C_2$$

$$r = R_0, \theta = \frac{\pi}{2} \Rightarrow C_2 = R_0 \Rightarrow r = R_0 \sin^2\theta$$

F

Question: For the field line in (e), if $R_0 = 2R$, at what angles of θ does the field line contact the sphere?

Solution:

$$R_0 = 2R \Rightarrow \frac{R}{R_0} = \frac{1}{2} = \sin^2\theta$$

$$\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4} \ (45^\circ, 135^\circ)$$

Problem 3

A reluctance motor is made by placing a high permeability material, which is free to rotate, in the air gap of a magnetic circuit excited by a current $i(t)$.

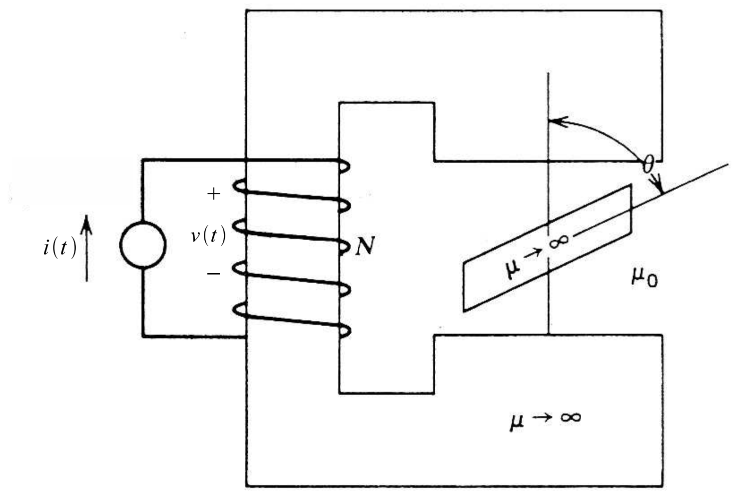

Figure 3: A diagram of a reluctance motor (Image by MIT OpenCourseWare).

The inductance of the magnetic circuit varies with rotor angle θ as

$$L(\theta) = L_0 + L_1 \cos 2\theta, L_0 > 0, 0 < L_1 < L_0$$

where the maximum inductance $L_0 + L_1$ occurs when $\theta = 0$ or $\theta = \pi$ and the minimum inductance $L_0 - L_1$ occurs when $\theta = \pm\frac{\pi}{2}$.

A

Question: What is the magnetic torque, T_{mag}, on the rotor as a funcion of the angle θ and current $i(t)$?
Solution:

$$T_{mag} = \frac{1}{2}i^2\frac{dL(\theta)}{d\theta} = \frac{1}{2}i^2(-L_1 2\sin 2\theta) = -L_1 i^2 \sin 2\theta$$

B

Question: With $i(t)$ a DC current I, a constant positive mechanical stress $T_{mech} > 0$ is applied. What is the largest value of $T_{mech} = T_{max}$ for which the rotor can be in static equilibrium?
Solution:

$$T_{mech} + T_{mag} = T_{mech} - L_1 i^2 \sin 2\theta = 0$$

maximum of $\sin 2\theta = 1$

$$T_{mech} = T_{max} = L_1 I^2$$

C

Question: If $T_{mech} = \frac{1}{2}T_{max}$, plot the total torque $T_{mag} + T_{mech}$. Use a graphical method to determine the equilibrium values of θ and label which are stable and which are unstable.

Solution:

$$T_T = T_{mech} + T_{mag} = L_1 I^2 \left(\frac{1}{2} - \sin 2\theta\right) = 0$$

$$\sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \; (30^\circ, 150^\circ)$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12} \; (15^\circ, 75^\circ, 195^\circ, 255^\circ)$$

$$\text{Stable if } \left.\frac{\partial T_T}{\partial \theta}\right|_{T_T=0} < 0 \quad \theta = 15^\circ, 195^\circ \text{ Stable}$$

$$\text{Unstable if } \left.\frac{\partial T_T}{\partial \theta}\right|_{T_T=0} > 0 \quad \theta = 75^\circ, 255^\circ \text{ Unstable}$$

D

Question: If the rotor has moment of inertia J and is slightly perturbed from a stable equilibrium position θ_{eq} at $t = 0$ by an angle position $\theta'(t)$, what is the general frequency of oscillation? What is the oscillation frequency for θ_{eq} found for stable equilibrium in part (c)?

Solution:

$$\theta = \theta_{eq} + \theta'(t)$$

$$T_T(\theta) = T_{mech} + T_{mag}(\theta = \theta_{eq}) = 0$$

$$T_T(\theta = \theta_{eq} + \theta'(t)) = \cancelto{0}{T_T(\theta = \theta_{eq})} + \left.\frac{dT_T}{d\theta}\right|_{\theta=\theta_{eq}}^{(\theta'(t))}$$

$$\frac{Jd^2\theta}{dt^2} = \frac{Jd^2\theta'}{dt^2} = \left.\frac{dT_T}{d\theta}\right|_{\theta=\theta_{eq}} \theta'$$

$$\frac{d^2\theta'}{dt^2} - \frac{1}{J}\left.\frac{dT_T}{d\theta}\right|_{\theta=\theta_{eq}} \theta' = 0$$

$$\text{Let } \omega_0^2 = -\frac{1}{J}\left.\frac{dT_T}{d\theta}\right|_{\theta_{eq}} \Rightarrow \frac{d^2\theta'}{dt^2} + \omega_0^2\theta' = 0$$

$$\omega_0^2 > 0 \text{ if } \left.\frac{dT_T}{d\theta}\right|_{\theta_{eq}} < 0$$

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% Mathematica code for generating the graph
In[1] = f[angle_] = .5 - Sin[2*angle*2*Pi/360]
Out[1] = 0.5 - Sin[angle*Pi/90]

In[2] = Plot[f[angle],{angle,0,360},AxesLabel -> {"Angle Theta in Degrees", "Total Torque/(L1*I^2)"}]
Out[2] = %See Figure 4
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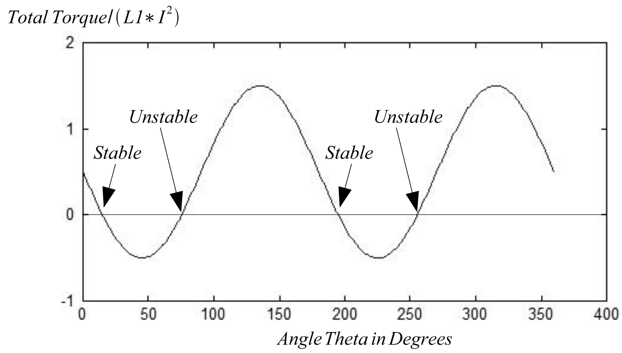

Figure 4: A graph plotting Total Torque versus Angle Theta in Degrees (Image by MIT OpenCourseWare).

$$\theta'(t) = A_1 \sin \omega_0 t + A_2 \cos \omega_0 t$$

Perturbations are bounded if ω_0 real ($\omega_0^2 > 0$)
This requires $\frac{dT_T}{d\theta}\Big|_{\theta_{eq}} < 0$

$$\frac{dT_T}{d\theta}\Bigg|_{\theta_{eq}} = -2L_1 I^2 \cos 2\theta_{eq} \Rightarrow \cos 2\theta_{eq} > 0 \text{ for stability}$$

θ_{eq}	$\cos 2\theta_{eq}$	Stability
15°	$\sqrt{3}/2$	Stable
75°	$-\sqrt{3}/2$	Unstable
195°	$\sqrt{3}/2$	Stable
255°	$-\sqrt{3}/2$	Unstable

E

Question: If the initial conditions of the perturbation are $\frac{d\theta'}{dt}\Big|_{t=0}$ and $\theta'(t = 0) = \Delta\theta$ what is $\theta'(t)$ for $t > 0$. Neglect any damping.
Solution:

$$\frac{d\theta'}{dt}\Bigg|_{t=0} = \omega_0(A_1 \cos \omega_0 t - A_2 \sin \omega_0 t)|_{t=0} = A_1\omega_0 = 0$$

$$A_1 = 0$$

$$\theta(t = 0) = \Delta\theta = A_2 \Rightarrow \theta(t) = \Delta\theta \cos \omega_0 t$$

$$\omega_0 = \left[\frac{1}{J} 2L_1 I^2\right]^{1/2}$$

F

Question: If $i(t)$ is a DC current I and a motor drives the rotor angle θ at constant angular speed Ω so that $\theta = \Omega t$, what is the voltage $v(t)$ across the coil?
Solution:

$$v(t) = \frac{d\lambda}{dt} = \frac{d[L(\theta)I]}{dt} = I\frac{dL(\theta)}{dt} = I\frac{dL(\theta)}{d\theta}\frac{d\theta}{dt}$$

$$v(t) = -2L_1 I\Omega \sin 2\Omega t$$

Problem 4

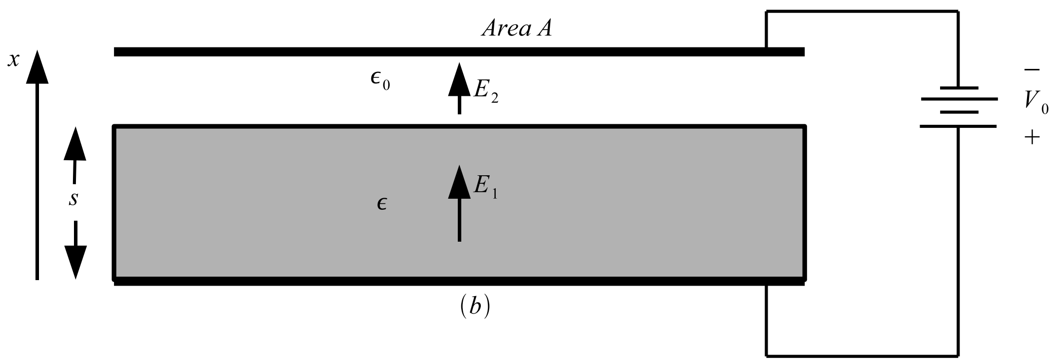

Figure 5: A diagram of a parallel plate capacitor with two dielectrics in series (Image by MIT OpenCourseWare).

A parallel plate capacitor with electrodes of area A has its upper electrode in a free space region in series with a solid dielectric of thickness s and dielectric permittivity ϵ. The $x = s$ interface has no free surface charge.

A

Question: What are the electric fields E_1 and E_2 in the dielectric and free space regions?
Solution:

$$V_0 = E_1 s + E_2(x - s)$$

$$\epsilon E_1 = \epsilon_0 E_2 \Rightarrow E_2 = \frac{\epsilon}{\epsilon_0} E_1$$

$$E_1\left[s + \frac{\epsilon}{\epsilon_0}(x - s)\right] = V_0$$

$$E_1 = \frac{\epsilon_0 V_0}{[\epsilon_0 s + \epsilon(x - s)]}$$

$$E_2 = \frac{\epsilon V_0}{[\epsilon_0 s + \epsilon(x - s)]}$$

B

Question: What is the free surface charge density on the lower electrode?
Solution:

$$\sigma_s(x=0) = \epsilon E_1 = \frac{\epsilon\epsilon_0 V_0}{\epsilon_0 s + \epsilon(x-s)}$$

C

Question: What is the capacitance $C(x)$ of the capacitor?
Solution:

$$C(x) = \frac{\sigma_s A}{V_0} = \frac{\epsilon\epsilon_0 A}{\epsilon_0 s + \epsilon(x-s)}$$

D

Question: what is the electric force on the upper electrode?
Solution:

$$f_x = \frac{1}{2}V_0^2\frac{dC(x)}{dx} = -\frac{1}{2}\frac{V_0^2\epsilon^2\epsilon_0 A}{[\epsilon_0 s + \epsilon(x-s)]^2}$$