6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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Additional Problems - Solutions

Prof. Markus Zahn | MIT OpenCourseWare

Problem 1.1

I

A

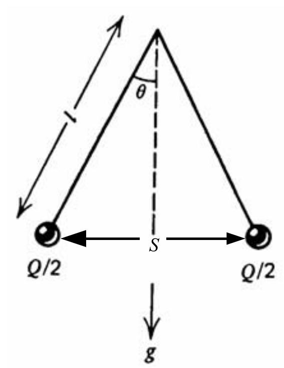

Figure 1: A diagram of two identical conducting balls suspended by essentially weightless strings of length l.

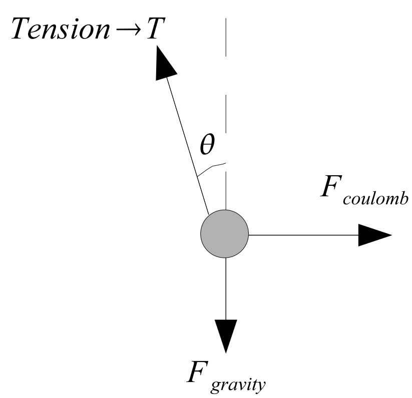

Figure 2: A diagram showing the forces on one of the balls (Image by MIT OpenCourseWare).

$$T = \frac{F_{gravity}}{\cos\theta} = \frac{F_{coulomb}}{\sin\theta} \Rightarrow \frac{Mg}{\cos\theta} = \frac{\frac{Q}{2}\frac{Q}{2}}{4\pi\epsilon_0 s^2 \sin\theta}$$

$$\frac{Mg4\pi\epsilon_0 s^2 \sin\theta}{\frac{Q^2}{4}\cos\theta} = 1 \Rightarrow \tan\theta = \frac{Q^2}{16\pi\epsilon_0 s^2 M}$$

$$\sin\theta = \frac{\frac{s}{2}}{l} = \frac{s}{2l} \Rightarrow \tan\theta \sin^2\theta = \frac{Q^2}{16\pi\epsilon_0 s^2 Mg}\left(\frac{s}{2l}\right)^2 = \frac{Q^2}{64\pi\epsilon_0 l^2 Mg}$$

II

A

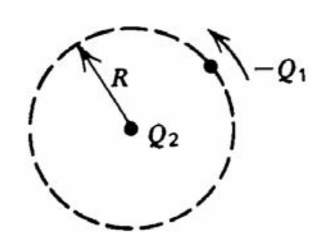

Figure 3: A diagram showing two charges, one at the center of another orbiting charge.

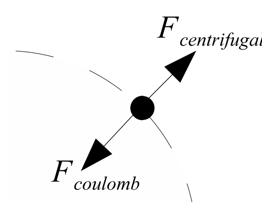

Figure 4: A diagram showing the forces on the orbiting charge (Image by MIT OpenCourseWare).

$$F_{centrifugal} = F_{coulomb}$$

$$m\omega^2 R = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \Rightarrow \omega = \sqrt{\frac{Q_1 Q_2}{4\pi\epsilon_0 R^3 m}}$$

B

$$Q_1 = e \quad Q_2 = ze$$

$$L = mvr = m\omega R^2 = \frac{nh}{2\pi} \Rightarrow m^2\omega^2 R^4 = \left(\frac{nh}{2\pi}\right)^2 = \frac{m^2 R^4 Q_1 Q_2}{4\pi\epsilon_0 R^3 m} = \frac{m Q_1 Q_2 R}{4\pi\epsilon_0}$$

$$\Rightarrow R = \frac{4\pi\epsilon_0 \frac{n^2 h^2}{4\pi^2}}{m Q_1 Q_2} = \frac{\epsilon_0 h^2}{m e^2 z \pi} n^2$$

C

Hydrogen atom $\Rightarrow z = 1$

$$e = 1.6022 \times 10^{-19} \text{ C}$$

$$m = 9.1094 \times 10^{-31} \text{ kg}$$

$$\epsilon_0 = 8.8542 \times 10^{-12}\ \text{Fm}^{-1}$$

$$h = 6.6261 \times 10^{-34}\ \text{Js}$$

$$R_{min} = \frac{\epsilon_0 h^2}{me^2 1\pi} 1^2 = 5.2917 \times 10^{-11}\ \text{m} \approx 0.0529\ \text{nm}$$

$$mvR = \frac{nh}{2\pi} \Rightarrow v = \frac{1h}{2\pi m R_{min}} \cong 2.19 \times 10^6\ \text{m/s}$$

III

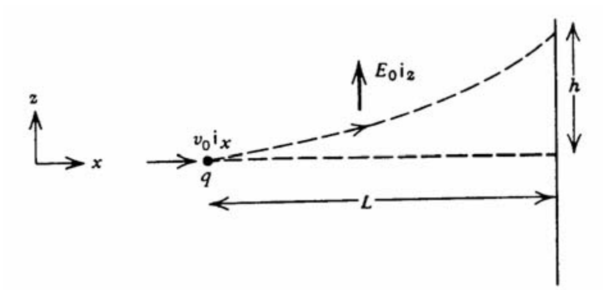

Figure 5: A diagram showing a charge q moving at velocity $v_0\ \mathbf{i_x}$ at a distance L from a screen in an electric field $E_0\mathbf{i_z}$ moving in a curved path until finally contacting the screen at $y = h$.

$$m\frac{d^2x}{dt^2} = 0 \Rightarrow \frac{dx}{dt} = v_0 \Rightarrow x = v_0 t \Rightarrow t = \frac{x}{v_0}$$

$$m\frac{d^2z}{dt^2} = qE_0 \Rightarrow \frac{dz}{dt} = \frac{q}{m}E_0 t + \cancelto{0}{v_{z_0}}$$

$$\Rightarrow z = \frac{qE_0}{2m}t^2 + \cancelto{0}{v_{z_0}}t + \cancelto{0}{z_0} \Rightarrow z = \frac{qE_0}{2m}t^2 = \frac{9E_0}{2m}\frac{x^2}{v_0^2}$$

$$\Rightarrow h = z(x = L) \Rightarrow h = \frac{9E_0 L^2}{2mv_0^2}$$

IV

A

$$\frac{1}{2}mv_x^2 - eV_1 = \frac{1}{2}mV_0^2 \Rightarrow v_x^2 = v_0^2 + \frac{2e}{m}V_1$$

B

From Lorentz force eqn: $E_y - v_x B_0 = 0 \Rightarrow v_x = \dfrac{E_y}{B_0} = \sqrt{v_0^2 + \dfrac{2e}{m}V_1}$

Since $E_y = \dfrac{V_2}{s} \Rightarrow \dfrac{V_2^2}{s^2 B_0^2} = v_0^2 + \dfrac{2e}{m}V_1 \Rightarrow v_0 = \sqrt{\dfrac{V_2^2}{s^2 B_0^2} - \dfrac{2eV_1}{m}}$

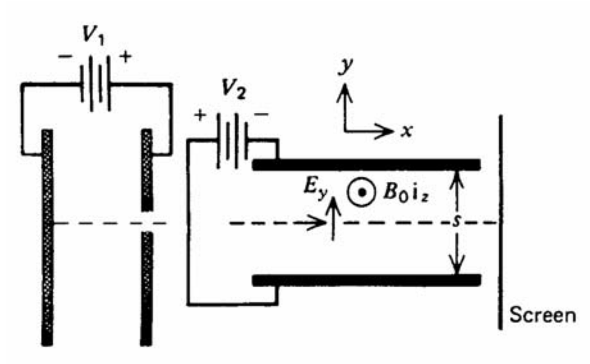

Figure 6: A diagram of a cathode-ray tube.

C

$$ev_x B_0 = \frac{mv_x^2}{R} \Rightarrow R = \frac{mv_x}{B_0 e} = \frac{m}{B_0 e}\sqrt{v_0^2 + \frac{2e}{m}V_1}$$

D

$$R = \frac{1}{B_0}\frac{m}{e}\sqrt{v_0^2 + \frac{2e}{m}V_1} \Rightarrow R_2 B_0^2 = \left(\frac{m}{e}\right)^2 v_0^2 + 2V_1\left(\frac{m}{e}\right)$$

$$\Rightarrow \left(\frac{m}{e}\right)^2 + \frac{2V_1}{V_0^2}\left(\frac{m}{e}\right) - \frac{R^2B_0^2}{V_0^2} = 0 \Rightarrow \frac{e}{m} = \frac{v_0^2}{\sqrt{V_1^2 + B_0^2R^2v_0^2} - V_1}$$

V

$$m\frac{d\vec{v}}{dt} = q\vec{v} \times (\mu_0 \vec{H}) \text{ where } \vec{H} = H_0\overline{i_z}, \qquad \vec{v} = v_x\overline{i_x} + v_y\overline{i_y} + v_z\overline{i_z}$$

$$\Rightarrow m\frac{d\vec{v}}{dt} = q\mu_0 \begin{vmatrix} \overline{i_x} & \overline{i_y} & \overline{i_z} \\ v_x & v_y & v_z \\ 0 & 0 & H_0 \end{vmatrix}$$

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{q\mu_0}{m}(v_y H_0 \overline{i_x} - v_x H_0 \overline{i_y})$$

$$\Rightarrow \frac{dv_x}{dt} = \frac{q\mu_0}{m}v_y H_0 \ (1), \quad \frac{dv_y}{dt} = -\frac{q\mu_0}{m}v_x H_0 \ (2), \quad \frac{dv_z}{dt} = 0 \ (3)$$

Taking the time derivative of (1) and using (2), get differential equation for v_x and v_y

$$\frac{d^2v_x}{dt^2} = \frac{q\mu_0 H_0}{m}\frac{dv_y}{dt} = \frac{-q^2\mu_0^2}{m^2}H_0^2 v_x$$

$$\frac{d^2v_y}{dt^2} = -\frac{q\mu_0 H_0}{m}\frac{dv_x}{dt} \Rightarrow \frac{d^2v_y}{dt^2} = -\frac{q^2\mu_0^2H_0^2}{m^2}v_y$$

$$\Rightarrow v_x = A\cos\left(\frac{q\mu_0 H_0}{m}t\right) + B\sin\left(\frac{q\mu_0 H_0}{m}t\right)$$

$$v_y = B\cos\left(\frac{q\mu_0 H_0}{m}t\right) - A\sin\left(\frac{q\mu_0 H_0}{m}t\right)$$

$$v_z = v_{z0}$$

Initial conditions:

$$v_x(t=0) = \boxed{v_{x0} = A}, \quad v_y(t=0) = \boxed{v_{y0} = B}$$

$$\Rightarrow \boxed{\begin{aligned} v_x(t) = &\quad v_{x0}\cos\left(\frac{q\mu_0 H_0}{m}t\right) + v_{y0}\sin\left(\frac{q\mu_0 H_0}{m}t\right) \\ v_y(t) = &\quad v_{y0}\cos\left(\frac{q\mu_0 H_0}{m}t\right) - v_{x0}\sin\left(\frac{q\mu_0 H_0}{m}t\right) \\ v_z(t) = &\quad v_{z0} \end{aligned}}$$

NOTE:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}, \quad \omega_0 = \frac{q\mu_0 H_0}{m}$$

$$= \Big(v_{x0}^2\cos^2(\omega_0 t) + v_{y0}^2\sin^2(\omega_0 t) + \cancel{2v_{x0}v_{y0}\sin(\omega_0 t)\cos(\omega_0 t)}$$

$$+v_{y0}^2\cos^2(\omega_0 t) + v_{x0}^2\sin^2(\omega_0 t) - \cancel{2v_{x0}v_{y0}\sin(\omega_0 t)\cos(\omega_0 t)} + v_{z0}^2\Big)^{1/2}$$

$$= \sqrt{v_{x0}^2\left(\cos^2(\omega_0 t) + \sin^2(\omega_0 t)\right) + v_{y0}^2\left(\cos^2(\omega_0 t) + \sin^2(\omega_0 t)\right) + v_{z0}^2}$$

$$= \sqrt{v_{x0}^2 + v_{y0}^2 + v_{z0}^2} \Rightarrow \text{ constant in time}$$

A

Velocity in xy-plane is $v_{xy} = \sqrt{v_{x0}^2 + v_{y0}^2}$

Centripetal acceleration is $\frac{v_{xy}^2}{r}$ where r is the radius of the circle

$$\underbrace{\left|m\frac{v_{xy}^2}{r}\right|}_{\substack{\text{force due to}\\ \text{centripetal}\\ \text{acceleration}}} = \underbrace{\left|q\vec{v}\times\mu_0\vec{H}\right|}_{\substack{\text{force caused}\\ \text{by magnetic}\\ \text{field}}} \Rightarrow m\frac{v_{xy}^2}{r} = qv_{xy}\mu_0 H_0 \Rightarrow r = \frac{mv_{xy}}{q\mu_0 H_0}$$

$$\Rightarrow \boxed{r = \frac{m}{q\mu_0 H_0}\sqrt{v_{x0}^2 + v_{y0}^2}}$$

B

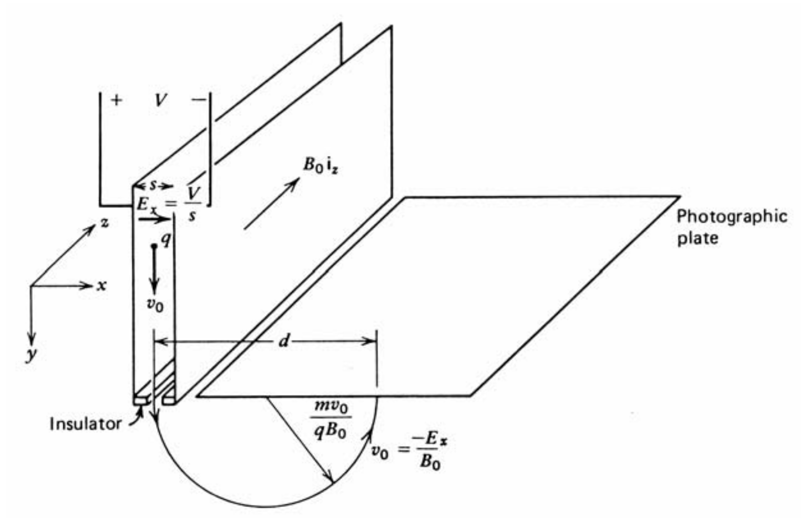

Figure 7: A diagram of a mass spectrograph.

$$\begin{aligned}\vec{f} = 0 &= q\vec{E} + q\vec{v} \times \mu_0 \vec{H} \\ &= q\vec{E} + q(v_0 \overline{i_y}) \times (\mu_0 H_0 \overline{i_z}) \\ &\Rightarrow \boxed{\begin{array}{l}\vec{E} = -v_0 \mu_0 H \overline{i_x} \\ V = -v_0 \mu_0 H_0 s\end{array}}\end{aligned}$$

C

$$d = 2r = 2\frac{mv_0}{qB_0} \text{ ; and from part (b): } v_0 = \frac{V}{\mu_0 H_0 s}$$

$$\Rightarrow \boxed{d = \frac{2mV}{qB_0^2 s}} \qquad \text{for} \quad \begin{array}{l} V = 100\text{Volts} \\ s = 1\text{cm} = 0.01\text{m} \\ \mu_0 H_0 = B_0 = 1\text{Tesla} \\ q = e = 1.6022 \times 10^{-19}\text{C} \\ m = 1.67 \times 10^{-27}\text{kg (mass of proton and neutron)} \end{array}$$

$$\Rightarrow \text{for Mg}^{24} \quad d = \frac{2 \times 24 \times 1.67 \times 10^{-27} \times 100}{1.6022 \times 10^{-19} \times 1 \times 0.01} = 5 \times 10^{-3} \approx 5.003\text{mm}$$

$$\Rightarrow \text{for Mg}^{25} \quad d \approx 5.2116\text{mm}$$

$$\Rightarrow \text{for Mg}^{26} \quad d \approx 5.42\text{mm}$$

Problem 1.2

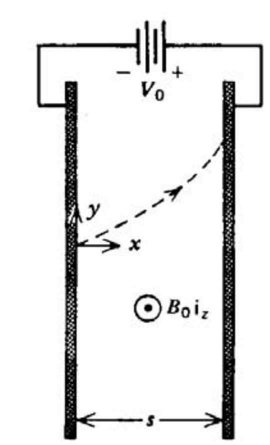

Figure 8: A diagram of two parallel plates connected to a power source creating a magnetron.

A

$$m\frac{dv_x}{dt} = -e\left(-\frac{V_0}{s} + v_y B_0\right)$$

$$m\frac{dv_y}{dt} = e v_x B_0 \Rightarrow v_x = \frac{m}{eB_0}\frac{dv_y}{dt} \Rightarrow \frac{dv_x}{dt} = \frac{m}{eB_0}\frac{d^2 v_y}{dt^2}$$

$$\Rightarrow \frac{d^2 v_y}{dt^2} + \omega_0^2 v_y = \frac{\omega_0^2 V_0}{B_0 s} \text{ where } \omega_0 = \frac{eB_0}{m}$$

$$\Rightarrow v_y = A_1 \sin(\omega_0 t) + A_2 \cos(\omega_0 t) + \frac{V_0}{B_0 s}$$

$$v_y(t=0) = 0 \Rightarrow A_2 = -\frac{V_0}{B_0 s}$$

$$v_x(t=0) = 0 \Rightarrow \left.\frac{dv_y}{dt}\right|_{t=0} = 0 \Rightarrow A_1 = 0$$

$$\Rightarrow \boxed{v_y = \tfrac{V_0}{B_0 s}\left(1 - \cos(\omega_0 t)\right)}\ ,\ v_x = \frac{1}{\omega_0}\frac{dv_y}{dt} = \boxed{\tfrac{V_0}{B_0 s}\sin(\omega_0 t) = v_x}$$

$$\boxed{x = \int v_x dt = \underbrace{\frac{V_0}{B_0 s \omega_0}\left(1 - \cos(\omega_0 t)\right)}_{\text{use B.C. } x|_{t=0} = 0}}\ ,\ \boxed{y = \int v_y dt = \underbrace{\frac{V_0}{B_0 s}\left(t - \frac{\sin(\omega_0 t)}{\omega_0}\right)}_{\text{use B.C. } y|_{t=0} = 0}}$$

B

$$x_{max} = \frac{2V_0}{B_0 s \omega_0} < s \Rightarrow \frac{2V_0}{B_0 s}\frac{m}{eB_0} < s \Rightarrow B_0^2 > \frac{2V_0 m}{es^2} \text{ for cut-off}$$

C

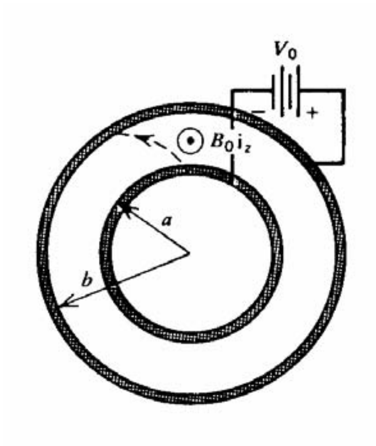

Figure 9: A diagram of two concentric conducting cylinders connected to a voltage source creating a magnetron.

Electrons injected from $r = a$, $\phi = 0$ with zero initial velocity

$$\overline{i_r} = \cos\phi\overline{i_x} + \sin\phi\overline{i_y}\ ,\ \overline{i_\phi} = -\sin\phi\overline{i_x} + \cos\phi\overline{i_y}$$

$$\frac{d\overline{i_r}}{dt} = \left[-\sin\phi\overline{i_x} + \cos\phi\overline{i_y}\right]\frac{d\phi}{dt} = \overline{i_\phi}\frac{d\phi}{dt} = \frac{v_\phi}{r}\overline{i_\phi}$$

$$\frac{d\overline{i_\phi}}{dt} = \left[-\cos\phi\overline{i_x} - \sin\phi\overline{i_y}\right]\frac{d\phi}{dt} = -\overline{i_r}\frac{v_\phi}{r}$$

$$\overline{v} = v_r\overline{i_r} + v_\phi\overline{i_\phi}$$

Acceleration

$$\overline{a} = \frac{d\overline{v}}{dt} = \overline{i_r}\frac{dv_r}{dt} + v_r\frac{d\overline{i_r}}{dt} + \overline{i_\phi}\frac{dv_\phi}{dt} + v_\phi\frac{d\overline{i_\phi}}{dt} = \overline{i_r}\frac{dv_r}{dt} + v_r\left(\frac{v_\phi}{r}\overline{i_\phi}\right) + \overline{i_\phi}\frac{dv_\phi}{dt} + v_\phi\left(-\frac{v_\phi}{r}\overline{i_r}\right)$$

$$\boxed{\overline{a} = \overline{i_r}\left(\frac{dv_r}{dt} - \frac{v_\phi^2}{r}\right) + \overline{i_\phi}\left(\frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r}\right)}$$

D

$$m\frac{d\overline{v}}{dt} = -e\left[\overline{E} + \overline{v}\times\overline{B}\right]\ ;\ \overline{E} = \frac{-V_0}{r\ln\frac{b}{a}}\overline{i_r}$$

$$\overline{i_r}\text{ component } \Rightarrow \left[\frac{dv_r}{dt} - \frac{v_\phi^2}{r}\right] = \frac{-eE_r}{m} - \frac{ev_\phi B_0}{m}$$

$$= \frac{eV_0}{mr\ln\frac{b}{a}} - \omega_0 v_\phi \quad \text{where } \omega_0 = \frac{eB_0}{m}$$

$$\overline{i_\phi} \Rightarrow \left[\frac{dv_\phi}{dt} + \frac{v_\phi v_r}{r}\right] = \frac{ev_r B_0}{m} = \omega_0 v_r$$

Use of hint:

$$\frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r \frac{dv_r}{dr} = \frac{d}{dr}\left(\frac{1}{2}v_r^2\right)$$

$$\frac{dv_\phi}{dt} = \frac{dv_\phi}{dr}\frac{dr}{dt} = v_r \frac{dv_\phi}{dr}$$

$\overline{i_\phi}$ component

$$\Rightarrow \left[v_r \frac{v_\phi}{dt} + \frac{v_\phi v_r}{r}\right] = v_r \underbrace{\left[\frac{dv_\phi}{dr} + \frac{v_\phi}{r}\right]}_{\frac{1}{r}\frac{d}{dr}(v_\phi r)} = \omega_0 v_r$$

$$\Rightarrow \frac{1}{r}\frac{d}{dr}(v_\phi r) = \omega_0$$

$$\Rightarrow r v_\phi = \frac{\omega_0 r^2}{2} + \text{ constant}$$

$$v_\phi(r=a) = 0 \Rightarrow \boxed{v_\phi = \frac{\omega_0}{2}\left(r - \frac{a^2}{r}\right)}$$

$\overline{i_r}$ component

$$\left[\frac{dv_r}{dt} - \frac{v_\phi^2}{r}\right] = \frac{d}{dr}\left(\frac{1}{2}v_r^2\right) - \frac{\omega_0^2}{4r}\left(r - \frac{a^2}{r}\right)^2 = \frac{eV_0}{mr\ln\frac{b}{a}} - \frac{\omega_0^2}{2}\left(r - \frac{a^2}{r}\right)$$

$$\Rightarrow \frac{d}{dr}\left(\frac{1}{2}v_r^2\right) = \frac{eV_0}{mr\ln\frac{b}{a}} - \frac{\omega_0^2}{4}\left(r - \frac{a^2}{r}\right)\left(1 + \frac{a^2}{r^2}\right) = \frac{eV_0}{rm\ln\frac{b}{a}} - \frac{\omega_0^2}{4}r\left(1 - \frac{a^4}{r^4}\right)$$

$$\Rightarrow \frac{1}{2}v_r^2 = \left[\frac{eV_0}{m\ln\frac{b}{a}}\right]\ln\frac{r}{a} - \frac{\omega_0^2}{4}\left[\frac{r^2}{2} + \frac{a^4}{2r^2} - a^2\right] \leftarrow \text{ using B.C. } v_r|_{(r=a)} = 0$$

$$= \frac{eV_0}{m\ln\frac{b}{a}}\ln\frac{r}{a} - \frac{\omega_0^2}{8r^2}\left[r^2 - a^2\right]^2$$

$$\boxed{v_r = \sqrt{\frac{2eV_0}{m\ln\frac{b}{a}}\ln\frac{r}{a} - \frac{\omega_0^2}{4r^2}\left[r^2 - a^2\right]^2}}$$

E

For cut-off $\Rightarrow v_r(r=b) < 0$

$$\Rightarrow \frac{2eV_0}{m\cancel{\ln\frac{b}{a}}}\cancel{\ln\frac{b}{a}} < \frac{\omega_0^2}{4b^2}\left(b^2 - a^2\right)^2$$

$$\Rightarrow \frac{2\cancel{e}V_0}{\cancel{m}} < \frac{e^{\cancel{2}}B_0^2}{4b^2m^{\cancel{2}}}\left(b^2 - a^2\right)^2$$

$$\boxed{\frac{8b^2mV_0}{e(b^2-a^2)^2} < B_0^2} \Leftarrow \text{ cut-off condition}$$

Check: cylindrical geometry approaches planar geometry of (a) if $b = a + s$ where $s << a$

$$\left(b^2 - a^2\right)^2 \rightarrow \left((a+s)^2 - a^2\right)^2$$
$$\rightarrow \left(a^2\left(1+\frac{s}{a}\right)^2 - a^2\right)^2$$
$$\rightarrow \left(a^2\left(1+\frac{2s}{a}\right) - a^2\right)^2$$
$$\rightarrow \left(\cancel{a^2} + 2as - \cancel{a^2}\right)^2$$
$$\rightarrow (2as)^2$$
$$b^2 \sim a^2$$

$$B_0^2 > \frac{8b^2 mV_0}{e\left(b^2 - a^2\right)^2}$$

for $s << a$

$$B_0^2 > \frac{8a^2 mV_0}{e\,(2as)^2} = \frac{8\cancel{a^2} mV_0}{e4\cancel{a^2}s^2}$$
$$B_0^2 > \frac{2mV_0}{es^2} \leftarrow \text{ agrees with (b)}$$

Problem 1.3

By problem:

$$\rho = \begin{cases} \frac{\rho_b r}{b}; & r < b \\ \rho_a; & b < r < a \end{cases}$$

Also, no σ_s at $r = b$, but non zero σ_s such that $\vec{E} = 0$ for $r > a$

A

By Gauss' Law:

$$\oint_{S_r} \epsilon_0 \vec{E} \cdot d\vec{a} = \int_{V_r} \rho dV \ ; \ S_R = \text{sphere with radius r}$$

As shown in class, symmetry ensures $\vec{E}$ has only radial component: $\vec{E} = E_r \hat{l_r}$

LHS of Gauss' Law:

$$\oint_{S_r} \epsilon_0 \vec{E} \cdot d\vec{a} = \int_0^{2\pi} \int_0^{\pi} \epsilon_0 \left(E_r \hat{l_r}\right) \cdot \underbrace{\left(r^2 \sin\theta d\theta d\phi \hat{i}_r\right)}_{d\vec{a} \text{ in spherical coordinates}}$$

$$= \underbrace{4\pi r^2}_{A} E_r \epsilon_0 \quad \text{where A is the surface area of a sphere of radius r}$$

RHS of Gauss' Law:

For r<b:

$$\int_{V_R}\rho dV = \underbrace{\int_0^r}_{r}\underbrace{\int_0^{2\pi}}_{\phi}\underbrace{\int_0^{\pi}}_{\theta}\frac{\rho_b r}{b}\underbrace{r^2\sin\theta d\theta d\phi dr}_{dV\text{ - diff. vol. element}}$$

$$= \frac{4}{4}\frac{\pi r^4}{b}\rho_b = \frac{\pi r^4\rho_b}{b}$$

For r>b & r<a: (b<r<a):

$$\int_{V_R}\rho dV = \int_0\int_0^{2\pi}\int_0^{\pi}\frac{\rho_b r}{b}r^2\sin\theta d\theta d\phi dr + \int_b^r\int_0^{2\pi}\int_0^{\pi}\rho_a r^2\sin\theta d\theta d\phi dr$$

$$= \frac{\cancel{4}\pi\rho_b b^3}{\cancel{4}} + \frac{4\pi\rho_a\left(r^3-b^3\right)}{3}$$

$$= \pi\rho_b b^3 + \frac{4}{3}\pi\rho_a\left(r^3-b^3\right) \quad b<r<a$$

B

Equating LHS and RHS

$$4\pi r^2 E_r\epsilon_0 = \begin{cases}\frac{\cancel{4}\pi r^4}{\cancel{4}b}\rho_b\ ; & r<b\\ \frac{\cancel{4}\pi\rho_b b^3}{\cancel{4}} + \frac{4\pi\rho_a(a^3-b^3)}{3}\ ; & b<r<a\end{cases}$$

$$\boxed{E_r = \begin{cases}\frac{r^2\rho_b}{4\epsilon_0 b}\ ; & r<b\\ \frac{b^3\rho_b}{4\epsilon_0 r^2} + \frac{\rho_a(r^3-b^3)}{3\epsilon_0 r^2}\ ; & b<r<a\end{cases}}$$

C

Again: $\hat{n}\cdot\left(\epsilon_0 E^a - \epsilon_0 E^b\right) = \sigma_s$

$$\vec{E}(r=a^+) = 0$$

By part (a):

$$E_r(r=a_-) = \frac{\rho_b b^3}{4\epsilon_0 a^2} + \frac{\rho_a\left(a^3-b^3\right)}{3\epsilon_0 a^2}$$

$$\sigma_s = \hat{i}_r\cdot\left(-\epsilon_0\vec{E}(r=a^-)\right)\ ,\ \text{So:}$$

$$\sigma_s = -\left[\frac{\rho_b b^3}{4\epsilon_0 a^2} + \frac{\rho_a\left(a^3-b^3\right)}{3\epsilon_0 a^2}\right]$$

D

$$\begin{array}{llll} r < b & Q_b = \pi b^3 \rho_b & Q_\sigma(r=a) = \sigma_s 4\pi a^2 = -4\pi a^2 \left[\dfrac{\rho_b b^3}{4\epsilon_0 a^2} + \dfrac{\rho_a \left(a^3 - b^3\right)}{3\epsilon_0 a^2} \right] \\ b < r < a & Q_a = \frac{4}{3}\pi \left(a^3 - b^3\right) \rho_a & Q_\sigma = Q_b + Q_a + Q_\sigma = 0 \end{array}$$

Problem 1.4

A

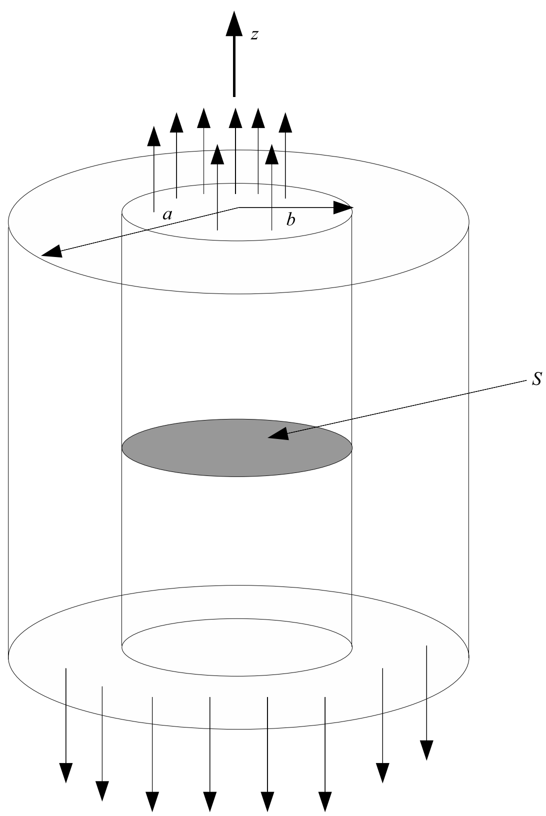

Figure 10: A diagram of a wire with z directed volume current with $-z$ directed surface current at $r = a$. (Image by MIT OpenCourseWare).

We are told current in $+z$ direction inside cylinder $r < b$

Current going through cylinder:

$$= I_{total} = \int_S \vec{J} \cdot d\vec{a} = \int_0^b \int_0^{2\pi} \underbrace{\left(\frac{J_0 r}{b} \hat{i}_z \right)}_{\vec{J}} \cdot \underbrace{\left(r d\phi dr \hat{i_z} \right)}_{d\vec{a}} = \frac{J_0 2\pi b^2}{3}$$

$$|\vec{K}| = \frac{\text{Total current in sheet}}{\text{length of sheet } (i.e., \text{ circumference of circle of radius a})}$$

Thus, $\vec{K}$'s units are $\frac{\text{Amps}}{\text{m}}$, whereas $\vec{J}$'s units are $\frac{\text{Amps}}{\text{m}^2}$

$$|\vec{K}| = \frac{\frac{2}{3}J_0\pi b^2}{2\pi a} = \frac{J_0 b^2}{3a}$$

$$\boxed{\vec{K} = -\frac{J_0 b^2}{3a}\hat{i_z}}$$

B

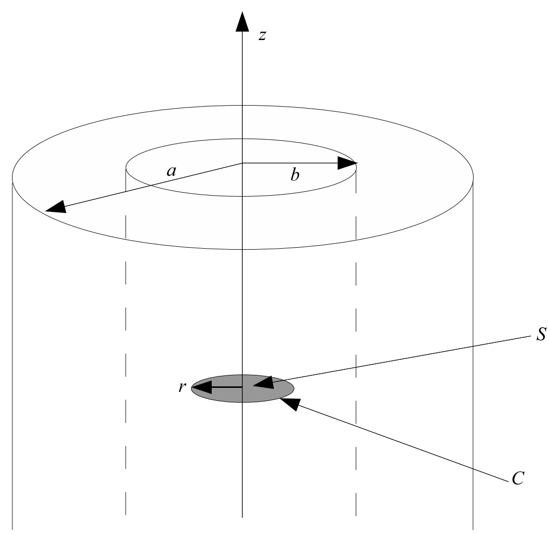

Figure 11: A diagram of the current carrying wire with a contour circle C centered on the z-axis with $r < b$ (Image by MIT OpenCourseWare).

Ampere's Law

$$\oint_C \vec{H}\cdot d\vec{s} = \int_S \vec{J}\cdot d\vec{a} + \underbrace{\frac{d}{dt}\int_S \epsilon_0\vec{E}\cdot d\vec{a}}_{\text{no } \vec{E} \text{ field, term is 0}}$$

Choose contour C as a circle and S as the minimum surface that the contour bounds (as shown in Figure 11)

Now solve LHS of Ampere's Law

$$\oint_C \vec{H}\cdot d\vec{s} = \int_0^{2\pi} \underbrace{(H_\phi\hat{i_\phi})}_{\vec{H}}\cdot\underbrace{(rd\phi)\hat{i_\phi}}_{d\vec{s}} = 2\pi r H_\phi$$

We assumed $H_z = H_r = 0$. This follows from the symmetry of the problem. $H_r = 0$ because $\oint_S \mu_0 \vec{H} \cdot d\vec{a} = 0$. In particular choose S as shown in Figure 12a.

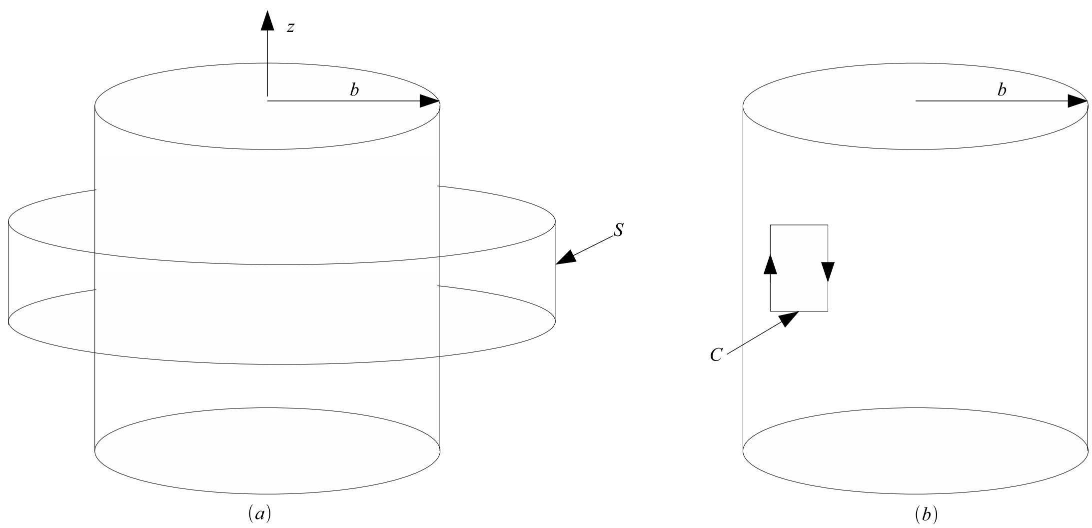

Figure 12: A diagram of the wire with the choice for S as well as a diagram of the wire with the choice of contour C (Image by MIT OpenCourseWare).

H_z is more difficult to see. It is discussed in Haus & Melcher, Chapter 1. The basic idea is to use the contour, C (depicted in Figure 12b), to show that if $H_z \neq 0$ it would have to be nonzero even at ∞, which is not possible without sources at ∞.

Now for RHS of Ampere's Law:

$\underline{r < b}$

$$\int_S \vec{J} \cdot d\vec{a} = \int_0^{2\pi} \int_0^r \underbrace{\left(\frac{J_0 r'}{b} \hat{i_z} \right)}_{\vec{J}} \cdot \underbrace{\left(r' dr' d\phi \hat{i_z} \right)}_{d\vec{a}}$$

$$= \frac{2 J_0 r^3 \pi}{3b}$$

$\underline{a > r > b}$

$$\int_S \vec{J} \cdot d\vec{a} = \int_0^{2\pi} \int_0^b \left(\frac{J_0 r'}{b} \hat{i_z} \right) \cdot \left(r' dr' d\phi \hat{i_z} \right) + \underbrace{\int_0^{2\pi} \int_b^r \left(0 \cdot \hat{i_z} \right) \cdot \left(r' dr' d\phi \hat{i_z} \right)}_{0}$$

$$= \frac{2}{3} J_0 b^2 \pi$$

Equating LHS & RHS:

$$2\pi r H_\phi = \begin{cases} \frac{2}{3b} J_0 r^3 \pi \ ; & r < b \\ \frac{2}{3} J_0 b^2 \pi \ ; & a > r > b \\ 0 \ ; & r > a \end{cases}$$

$$\boxed{\vec{H} = \begin{cases} \dfrac{J_0 r^2}{3b}\hat{i_\phi} \ ; & r < b \\ \dfrac{J_0 b^2}{3r}\hat{i_\phi} \ ; & a > r > b \\ 0 \ ; & r > a \end{cases}}$$

Problem 1.5

Demos 1.3.1, 1.5.1 Coulombs's Force Law and Measurements of Charge

- Rubbing of inflated balloons with a dry cloth
  - Accumulation of charge on balloon surfaces
  - Balloons repel each other because they have been charged to same polarity
  - Charges on balloons induce image charges of opposite polarity on conducting surfaces
    - Balloons are then attracted to these surfaces
- If we insert balloons in a Faraday cage
  - We can measure the charges on the balloons
  - It makes no difference to the measurement if balloons make contact with the inner surface
  - If balloon is broken in the Faraday cage the charge is not removed when the broken balloon pieces are removed

Demo 11.7.1: Steady state magnetic levitation

- Demonstrates magnetic forces due to conduction currents
- A pancake coil is excited by 60 Hz current and placed on an aluminum ground plane
- Typical currents of 20-30 Amps
- As current is raised, Lorentz force can overcome the coils weight and the pancake coil rises
- For ground plate thickness $\sim$ skin depth, the coil rises
- For ground plate thickness less than skin depth, the coil lifts up at higher current
- For ground plate thickness much less than skin depth, the coil does not lift up, because most of the magnetic field penetrates through the ground plane.

Problem 2.1

A

The idea here is similar to applying the chain rule in a 1D problem

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \left[\frac{d}{df}\left(\frac{1}{f(x)}\right)\right]\left[\frac{df}{dx}\right] = \frac{-f^{'}(x)}{f^2(x)}$$

$f(x)$ corresponds to $|\overline{r} - \overline{r}'|$. So, by diff. $f(x)$ we get part of the answer to the derivative of $\frac{1}{f(x)}$. But we can just do it directly too.

$$|\overline{r} - \overline{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\nabla\left[\frac{1}{|\overline{r} - \overline{r}'|}\right] = \hat{i}_x \frac{\partial}{\partial x}\left[\frac{1}{|\overline{r} - \overline{r}'|}\right] + \hat{i}_y \frac{\partial}{\partial y}\left[\frac{1}{|\overline{r} - \overline{r}'|}\right] + \hat{i}_z \frac{\partial}{\partial z}\left[\frac{1}{|\overline{r} - \overline{r}'|}\right]$$

So we can apply the trick above by just considering x, y, and z components separately.

$$\frac{\partial}{\partial x}|\overline{r} - \overline{r}'| = \frac{\partial}{\partial x}\left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)$$

$$= \frac{x - x'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{x - x'}{|\overline{r} - \overline{r}'|}$$

Similarly, $\frac{\partial}{\partial y}|\overline{r} - \overline{r}'| = \frac{y - y'}{|\overline{r} - \overline{r}'|}$ and $\frac{\partial}{\partial z}|\overline{r} - \overline{r}'| = \frac{z - z'}{|\overline{r} - \overline{r}'|}$.

$$\frac{\partial}{\partial x}\left(\frac{1}{|\overline{r} - \overline{r}'}\right) = \frac{-\frac{\partial}{\partial x}|\overline{r} - \overline{r}'|}{|\overline{r} - \overline{r}'|^2}$$

and so on for y and z.

$$|\overline{r} - \overline{r}'|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

so:

$$\nabla\left[\frac{1}{|\overline{r} - \overline{r}'|}\right] = \frac{-\left[(x-x')\hat{i}_x + (y-y')\hat{i_y} + (z-z')\hat{i_z}\right]}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$$

Denominators $= |\overline{r} - \overline{r}'|^{3/2}$. Thus,

$$\nabla\left[\frac{1}{|\overline{r} - \overline{r}'|}\right] = \frac{-(\overline{r} - \overline{r}')}{|\overline{r} - \overline{r}'|^3} = \frac{-1}{|\overline{r} - \overline{r}'|^2}\frac{(\overline{r} - \overline{r}')}{|\overline{r} - \overline{r}'|}$$

$$= \frac{-\hat{i}_{r'r}}{|\overline{r} - \overline{r}'|^2}$$

B

Follows from (A) immediately by substitution. Remember ∇ is derived in terms of unprimed x, y, z. ∇ does not affect x', y', z'.

C

$$\Phi(\overline{r}) = \int_{V'} \frac{\rho(\overline{r}')dV'}{4\pi\varepsilon_0|\overline{r}-\overline{r}'|}$$

$\rho(\overline{r}')$ = charge density in $\frac{C}{m^3}$. We have λ in units of $\frac{C}{m}$. In this sense, $\rho \to \infty$ at the ring. We can represent this in cylindrical coordinates by $\rho(\overline{r}') = \lambda_0\delta(z)\delta(r-a)$. Then we can evaluate the triple integral

$$\int\int\int \frac{\lambda_0\delta(z)\delta(r-a)rdrd\phi dz}{4\pi\epsilon_0|\overline{r}-\overline{r}'|} = \int_0^{2\pi} \frac{\lambda_0 ad\phi}{4\pi\epsilon_0|\overline{r}-\overline{r}'|}$$

But, we can skip that unnecessary work by simply considering infinitesimal charges $(ad\phi)\lambda_0$ around the ring.

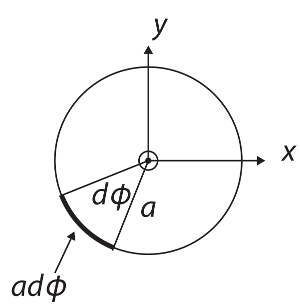

Figure 13: A ring of line charge with infinitesimal charge elements $qd = \lambda_0 ad\phi$. (Image by MIT OpenCourseWare).

We only care about z axis in this problem as well, so, by symmetry, there is no field in the x and y directions.

$$\Phi(\overline{r}) = \int_0^{2\pi} \frac{\lambda_0(ad\phi)}{4\pi\varepsilon_0 \underbrace{(a^2+z^2)^{1/2}}_{\substack{\text{distance from}\\ \text{the charge}\\ \text{element } \lambda_0 ad\phi\\ \text{to the point } z\\ \text{on the } z\text{-axis}}}}$$

$$\Phi(\overline{r}) = \frac{\lambda_0 a}{2\varepsilon_0(a^2+z^2)^{1/2}}$$

on the z-axis.

$$\overline{E} = -\nabla\Phi(\overline{r}) = -\left(\hat{i}_x \overbrace{\frac{\partial}{\partial x}\Phi}^{0} + \hat{i}_y \overbrace{\frac{\partial}{\partial y}\Phi}^{0} + \hat{i}_z\frac{\partial}{\partial z}\Phi\right)$$

$$\overline{E} = -\hat{i}_z\frac{\partial}{\partial z}\left(\frac{\lambda_0 a}{2\varepsilon_0(a^2+z^2)^{1/2}}\right)$$

$$\overline{E} = \hat{i}_z \frac{a\lambda_0 z}{2\varepsilon_0(a^2+z^2)^{3/2}}$$

Using the equation from the Problem 2.2 Statement with z component only (symmetry) and with $\rho(\overline{r}')dV' \rightarrow \lambda_0 a d\phi$

$$E_z(z) = \int_0^{2\pi} \frac{\lambda_0 a d\phi \cos\theta}{4\pi\varepsilon_0(z^2+a^2)}, \quad \cos\theta = \frac{z}{(a^2+z^2)^{1/2}}$$

$$= \int_0^{2\pi} \frac{\lambda_0 a z}{(a^2+z^2)^{3/2}} \frac{d\phi}{4\pi\varepsilon_0}$$

$$= \frac{\lambda_0 a z}{2\varepsilon_0(a^2+z^2)^{3/2}}$$

Limit $|z| \rightarrow \infty$

$$\sqrt{a^2+z^2} \rightarrow |z|$$

$$\Phi(z) \approx \frac{\lambda_0 a}{2\varepsilon_0(a^2+z^2)^{1/2}} \approx \frac{2\pi\lambda_0 a}{4\pi\varepsilon_0|z|} \approx \frac{Q}{4\pi\varepsilon_0|z|}$$

$Q = 2\pi\lambda_0 a$ (total charge on loop). $\Phi(z)$ looks like potential from point charge in far field.

$$E_z = \frac{\lambda_0 a z}{2\varepsilon_0(a^2+z^2)^{3/2}} \approx \frac{\lambda_0 a z}{2\varepsilon_0|z|^3} = \begin{cases} \frac{Q}{4\pi\varepsilon_0|z|^2} & z>0 \\ \frac{-Q}{4\pi\varepsilon_0|z|^2} & z<0 \end{cases}$$

D

From (C), $\Phi = \frac{\lambda_0 r}{2\varepsilon_0(r^2+z^2)^{1/2}}$ for a ring of radius r. But now we have σ_0, not λ_0. How do we express λ_0 in terms of σ_0? Take a ring of width dr in the disk (see Figure 14). Total charge in the ring = $\underbrace{(r)(2\pi)}_{\text{circumference}}(dr)\sigma_0$.

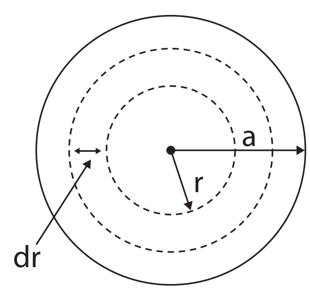

Figure 14: A line charge ring of width dr in the disk (Image by MIT OpenCourseWare).

Line charge density = $\lambda_0 = \frac{\text{total charge}}{\text{length}} = \sigma_0 dr$

So: $\lambda_0 = \sigma_0 dr$

$$d\Phi = \frac{\sigma_0 r dr}{2\varepsilon_0(r^2+z^2)^{1/2}}$$

$$\Phi_{\text{total}} = \int_0^a \frac{\sigma_0 r dr}{2\varepsilon_0(r^2+z^2)^{1/2}} = \frac{\sigma_0}{2\varepsilon_0}\int_0^a \frac{r dr}{(r^2+z^2)^{1/2}}$$

$$= \frac{\sigma_0}{2\varepsilon_0}\left[\sqrt{r^2+z^2}\right]\Big|_{r=0}^{r=a} = \frac{\sigma_0}{2\varepsilon_0}\left[\sqrt{a^2+z^2} - |z|\right]$$

$$\overline{E} = -\nabla\Phi_{\text{total}} = \frac{\sigma_0}{2\varepsilon_0} z\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{a^2+z^2}}\right]\overline{i_z}$$

E

As $z \to \infty$,

$$(a^2+z^2)^{1/2} \to |z| + \frac{a^2}{2|z|}; \quad (a^2+z^2)^{-1/2} \to \frac{1}{|z|}\left(1 - \frac{a^2}{2z^2}\right)$$

$$\Phi_{\text{total}} \to \frac{\pi a^2 \sigma_0}{4\epsilon_0 \pi |z|}$$

$$\bar{E} \to \frac{\pi a^2 \sigma_0}{4\pi\varepsilon_0 z^2}\overline{i_z}$$

just like a point charge of $\sigma_0 \pi a^2$.

F

As $a \to \infty$, z in the $\sqrt{a^2+z^2}$ can be neglected, so

$$\Phi_{\text{total}} \to \frac{\sigma_0}{2\varepsilon_0}\left[a - |z|\right]$$

$$E_z \to \frac{\sigma_0 z}{2\varepsilon_0}\left[\frac{1}{|z|} - 0\right] = \begin{cases} \frac{\sigma_0}{2\varepsilon_0} & z > 0 \\ \frac{-\sigma_0}{2\varepsilon_0} & z < 0 \end{cases}$$

just like a sheet charge.

G

For $\lambda(\phi) = \lambda_0 \sin\phi$

$$\Phi = \int_0^{2\pi} \frac{\lambda(\phi) a}{4\pi\epsilon_0\sqrt{a^2+z^2}} d\phi = \frac{a}{4\pi\epsilon_0\sqrt{a^2+z^2}} \int_0^{2\pi} \lambda_0 \sin\phi d\phi$$

$$= \frac{a\lambda_0}{4\pi\epsilon_0\sqrt{a^2+z^2}}(-\cos\phi)\Big|_0^{2\pi} = 0 \quad \text{along z axis}$$

The electric potential along the z axis is zero. It is not possible to find the electric field along the z-axis using the above result for the scalar electric potential value along the z-axis.

One cannot use Equation (2) from the problem statement to find $\overline{E}$ in this case.

H

$$\overline{E}(\overline{r}) = \int_{V'} \frac{\rho(\overline{r}')\overline{i_{r'r}}}{4\pi\epsilon_0|\overline{r}-\overline{r}'|^2} dV' = \int_{L'} \frac{\lambda(\overline{r}')\overline{i_{r'r}} dl'}{4\pi\epsilon_0|\overline{r}-\overline{r}'|^2}$$

The charge density along the hoop will have a sinusoidal variation as a function of ϕ as shown in Figure 16 with zero net charge on the hoop.

Due to odd symmetry of the charge distribution the z component of the $\overline{E}$ field will cancel out as shown in Figure 15.

$$i.e., \quad dE_z|_\phi = -dE_z|_{\phi+\pi}$$

Similarly due to symmetry with respect to the yz plane, the x component of the $\overline{E}$ field also cancels as shown in Figure 17.

Thus the resulting $\overline{E}$ field is in $-y$ direction only!

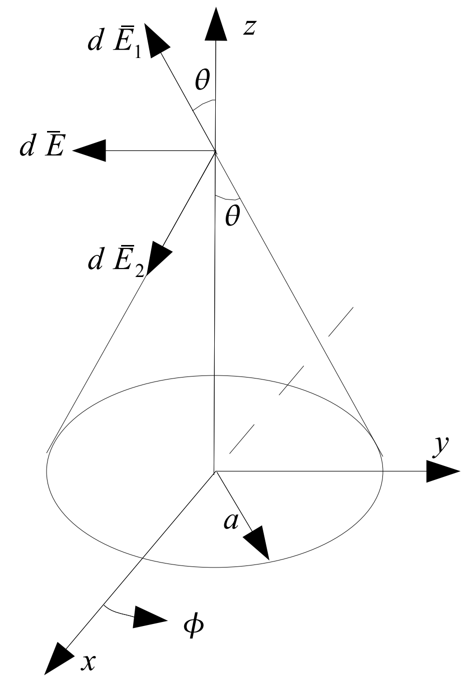

Figure 15: A diagram showing that the $\overline{E}$ field generated from the hoop of line charge with odd symmetry in angle ϕ is transverse to the z axis in the $-\overline{i_y}$ direction (Image by MIT OpenCourseWare).

$|d\overline{E}\sin\theta|$ gives the magnitude of the $\overline{E}$-field on the xy plane due to small line charge component $dq = \lambda a d\phi$

$$\lambda(\overline{r'}) = \lambda_0 \sin\phi$$

$$\overline{r} = z\overline{i_z}, \quad \overline{r}' = a\overline{i_r}$$

$$\overline{r} - \overline{r}' = z\overline{i_z} - a\overline{i_r}, \quad |\overline{r} - \overline{r}'| = \sqrt{z^2 + a^2}$$

$$\overline{i_{r'r}} = \frac{\overline{r} - \overline{r}'}{|\overline{r} - \overline{r}'|}, \quad dl' = ad\phi$$

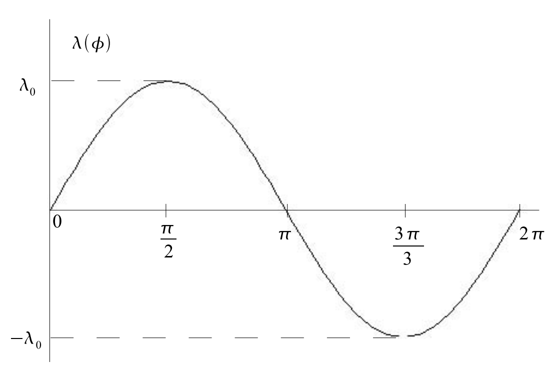

Figure 16: A graph showing the line charge density $\lambda_0 \sin\phi$ along the hoop as a function of ϕ (Image by MIT OpenCourseWare).

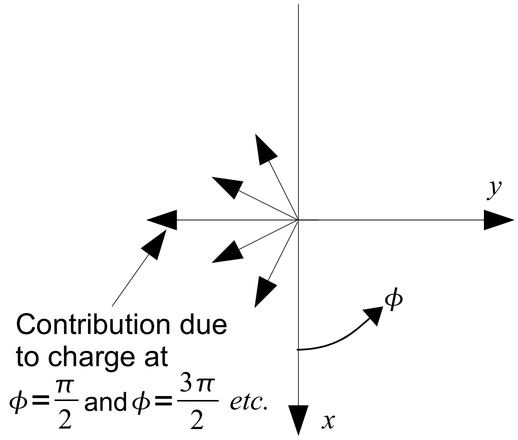

Figure 17: A diagram of the electric field components from hoop charge elements looking down along the z-axis showing that the x components cancel and the y components add in the $-y$ direction (Image by MIT OpenCourseWare).

$$\overline{E} = \int_{\phi=0}^{2\pi} \frac{\lambda_0 \sin\phi(z\overline{i_z} - a\overline{i_r})ad\phi}{4\pi\epsilon_0(z^2+a^2)^{3/2}}\ ,\ \overline{i_r} = \cos\phi\overline{i_x} + \sin\phi\overline{i_y}$$

$$E_x = \int_{\phi=0}^{2\pi} \frac{-\lambda_0 a^2 \sin\phi\cos\phi d\phi}{4\pi\epsilon_0(z^2+a^2)^{3/2}} = 0 \quad \left(\int_0^{2\pi}\sin\phi\cos\phi d\phi = 0\right)$$

$$E_y = \int_{\phi=0}^{2\pi} \frac{-\lambda_0 a^2 \sin^2\phi d\phi}{4\pi\epsilon_0(z^2+a^2)^{3/2}} = \frac{-\lambda_0 a^2}{4\epsilon_0(z^2+a^2)^{3/2}} \quad \left(\int_0^{2\pi}\sin^2\phi d\phi = \pi\right)$$

$$E_z = \int_{\phi=0}^{2\pi} \frac{\lambda_0 z \sin\phi a d\phi}{4\pi\epsilon_0(z^2+a^2)^{3/2}} = 0 \quad \left(\int_0^{2\pi}\sin\phi d\phi = 0\right)$$

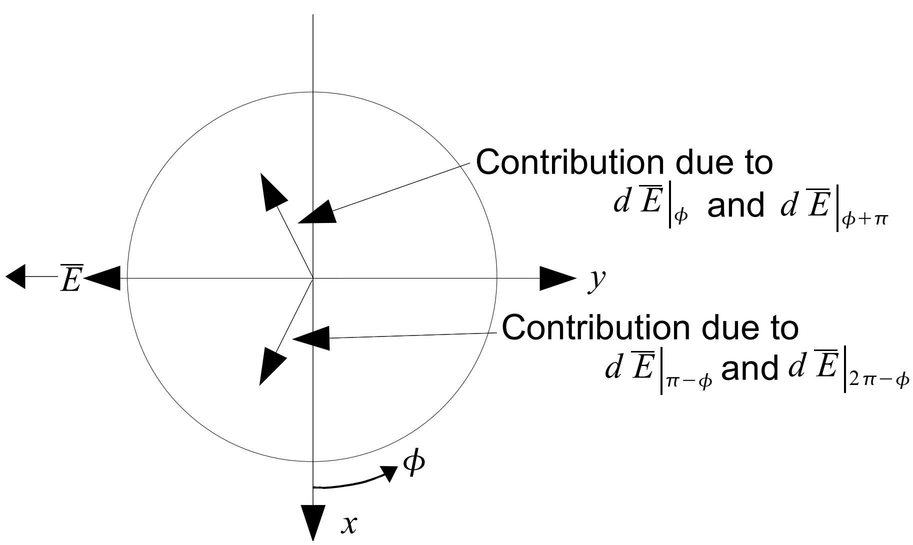

Figure 18: A diagram of the electric field components looking down along the z-axis (Image by MIT OpenCourseWare).

$$\overline{E} = -\frac{\lambda_0 a^2}{4\pi\epsilon_0}\frac{1}{(a^2+z^2)^{3/2}}\pi\overline{i_y} = \boxed{-\frac{\lambda_0 a^2}{4\epsilon_0(a^2+z^2)^{3/2}}\overline{i_y}}$$

$$\underbrace{\lim_{z\to\infty}}_{z>>a}\overline{E}(r=0,z) = \lim_{z\to\infty} -\frac{\lambda_0 a^2}{4\epsilon_0 z^3\left(1+\left(\frac{a}{z}\right)^2\right)^{3/2}}\overline{i_y} \approx \frac{-\lambda_0 a^2}{4\epsilon_0|x|^3}\overline{i_z} = \frac{-p_y}{4\pi\epsilon_0|z|^3}$$

$\overline{p} = \lambda_0\pi a^2\overline{i_y} \rightarrow$ acts like field due to a dipole charge with dipole moment p_y.

I

Surface charge density $\sigma(\phi) = \sigma_0 \sin\phi$

We can use the method of superposition using the result obtained in part (a)

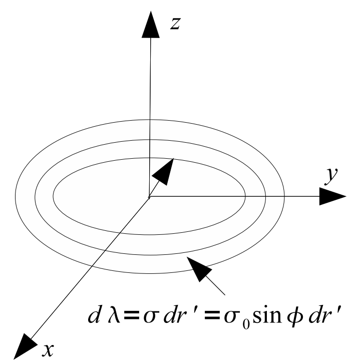

Figure 19: A diagram showing the method of superposition using discretized rings of line charge to approximate a surface charge distribution (Image by MIT OpenCourseWare).

$$\Phi = \int_0^a \int_0^{2\pi} \frac{\sigma(\phi)}{4\pi\epsilon_0\sqrt{(r')^2+z^2}} r' d\phi dr'$$

$$= \int_0^a \underbrace{\int_0^{2\pi} \frac{\sigma_0 \sin\phi}{4\pi\epsilon_0\sqrt{(r')^2+z^2}} d\phi r'}_{0} dr'$$

$= 0$ along z axis

So it is not possible to use Equation (2) in the problem statement to find the electric field along the z-axis.

J

Again using method of superposition with the result of part (b) for a hoop of radius r'

$$d\overline{E} = -\frac{d\lambda_0 (r')^2}{4\epsilon_0\left((r')^2+z^2\right)^{3/2}}\overline{i_y} = \frac{-\sigma(r')^2 dr'}{4\epsilon_0((r')^2+z^2)^{3/2}}\overline{i}_y$$

$$\Rightarrow \overline{E} = \int_{r'=0}^{a} -\frac{\sigma_0 (r')^2}{4\epsilon_0 \left((r')^2 + z^2\right)^{3/2}} \overline{i_y} dr'$$

$$= -\frac{\sigma_0}{4\epsilon_0}\overline{i_y}\left(-\frac{r'}{\sqrt{(r')^2+z^2}} + \ln\left[r' + \sqrt{(r')^2+z^2}\right]\right)\Bigg|_{r'=0}^{a}$$

$$= -\frac{\sigma_0}{4\epsilon_0}\overline{i_y}\left(-\frac{a}{\sqrt{a^2+z^2}} + \ln\left[a + \sqrt{a^2+z^2}\right] - \ln\left[\sqrt{z^2}\right]\right)$$

$$\boxed{\overline{E} = -\frac{\sigma_0}{4\epsilon_0}\overline{i_y}\left(-\frac{a}{\sqrt{a^2+z^2}} + \ln\left[\frac{a + \sqrt{a^2+z^2}}{|z|}\right]\right)}$$

$$= -\frac{\sigma_0}{4\epsilon_0}\left(\frac{-a|z|}{\sqrt{(\frac{a}{|z|})^2+1}} + \ln\left[\frac{a}{|z|} + \sqrt{(\frac{a}{|z|})^2+1}\right]\right)\bar{i}_y$$

$$= \frac{-\sigma_0}{4\epsilon_0}\frac{a^3}{3|z|^3}\bar{i}_y = \frac{-\sigma_0 a^3}{12\epsilon_0 |z|^3}\bar{i}_y$$

$$\frac{-p_y}{4\pi\epsilon_0 |z|^3} = \frac{-\sigma_0 a^3}{12\epsilon_0 |z|^3}$$

$$p_y = \frac{\sigma_0 \pi a^3}{3}$$

Problem 2.2

A

By the divergence theorem:

i

$$\int_V \nabla\cdot(\nabla\times\vec{A})dV = \oint_S (\nabla\times\vec{A})\cdot d\vec{a}$$

where S encloses V. By Stokes' Theorem:

ii

$$\int_{S'} (\nabla\times\vec{A})\cdot d\vec{a} = \oint_C \vec{A}\cdot d\vec{l}$$

Suppose S is as in Figure 20
and $S^{'}$ is as in Figure 21
i.e. $S^{'}$ is the same as S, except for the curve C, which makes $S^{'}$ slightly unclosed. Now consider limit as $C \to 0$ (Figure 22)

In limit $C \to 0, S^{'} \to S$. If C is 0, then $\oint_C \vec{A}\cdot d\vec{l} = 0$. By equation (ii), $\oint_S (\nabla\times\vec{A})\cdot d\vec{a} = 0$. By equation (i), $\int_V \nabla\cdot(\nabla\times\vec{A})dV = 0$. Since V can be any volume, argument of integral must be identically 0.

$$\nabla\cdot(\nabla\times\vec{A}) = 0$$

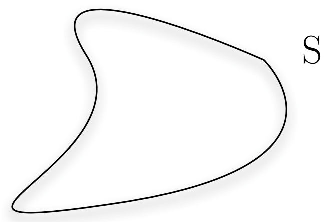

Figure 20: Closed surface S (Image by MIT OpenCourseWare).

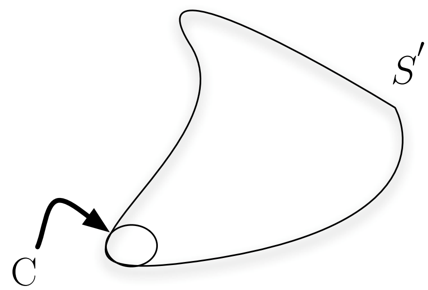

Figure 21: Open surface $S^{'}$ (Image by MIT OpenCourseWare).

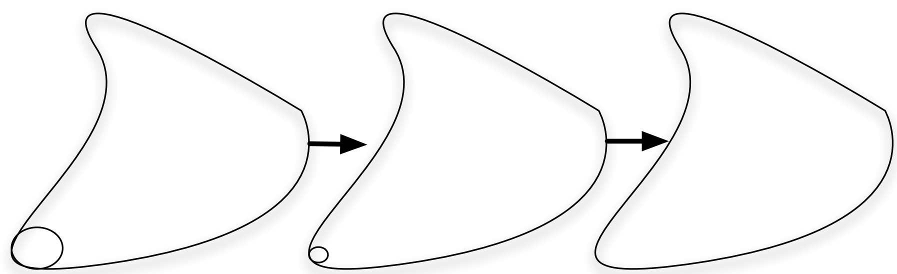

Figure 22: Limit as $C \to 0$ (Image by MIT OpenCourseWare).

B

$$\vec{A} = A_x\vec{i_x} + A_y\vec{i_y} + A_z\vec{i_z} \qquad \nabla = \vec{i_x}\frac{\partial}{\partial x} + \vec{i_y}\frac{\partial}{\partial y} + \vec{i_z}\frac{\partial}{\partial z}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i_x} & \vec{i_y} & \vec{i_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{i_x}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \vec{i_y}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) + \vec{i_z}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$$

$$\nabla \cdot (\nabla \times \vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = 0 \quad \text{using interchangability of partial derivatives}$$

$\nabla \cdot (\nabla \times \vec{A}) = 0$ in spherical coordinates

$$\vec{A} = A_r\vec{i_r} + A_\theta\vec{i_\theta} + A_\phi\vec{i_\phi}$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}\left(2rA_r + r^2\frac{\partial A_r}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\cos\theta A_\theta + \sin\theta\frac{\partial A_\theta}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$= \frac{2}{r}A_r + \frac{\partial A_r}{\partial r} + \frac{1}{r}\cot\theta A_\theta + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\vec{A} = \vec{i_r}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right) + \vec{i_\theta}\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right) + \vec{i_\phi}\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)$$

$$= \vec{i_r}\left[\frac{1}{r\sin\theta}\left(\cos\theta A_\phi + \sin\theta\frac{\partial A_\phi}{\partial\theta}\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right] + \vec{i_\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\left(A_\phi + r\frac{\partial A_\phi}{\partial r}\right)\right] +$$

$$+\vec{i_\phi}\left[\frac{1}{r}\left(A_\theta + r\frac{\partial A_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]$$

$$= \vec{i_r}\left[\frac{1}{r}\cot\theta A_\phi + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right] + \vec{i_\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}A_\phi - \frac{\partial A_\phi}{\partial r}\right] + \vec{i_\phi}\left[\frac{1}{r}A_\theta + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]$$

$$\Rightarrow \nabla\cdot(\nabla\times\vec{A}) = \frac{2}{r}\left[\frac{1}{r}\cot\theta A_\phi + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right] + \frac{\partial}{\partial r}\left[\frac{1}{r}\cot\theta A_\phi + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}\right]$$

$$+\frac{1}{r}\cot\theta\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}A_\phi - \frac{\partial A_\phi}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}A_\phi - \frac{\partial A_\phi}{\partial r}\right]$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\frac{1}{r}A_\theta + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right]$$

$$= \underbrace{\frac{2}{r^2}\cot\theta A_\phi}_{\text{VII}} + \underbrace{\frac{2}{r^2}\frac{\partial A_\phi}{\partial\theta}}_{\text{IV}} - \underbrace{\frac{2}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\phi}}_{\text{III}} - \underbrace{\frac{1}{r^2}\cot\theta A_\phi}_{\text{VII}} + \underbrace{\frac{1}{r}\cot\theta\frac{\partial A_\phi}{\partial r}}_{\text{II}} - \underbrace{\frac{1}{r^2}\frac{\partial A_\phi}{\partial\theta}}_{\text{IV}}$$

$$+\underbrace{\frac{1}{r}\frac{\partial^2 A_\phi}{\partial r\partial\theta}}_{\text{VI}} + \underbrace{\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial A_\theta}{\partial\phi}}_{\text{III}} - \underbrace{\frac{1}{r\sin\theta}\frac{\partial^2 A_\theta}{\partial r\partial\phi}}_{\text{V}} + \underbrace{\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_r}{\partial\phi}}_{\text{I}} - \underbrace{\frac{1}{r^2}\cot\theta A_\phi}_{\text{VII}} - \underbrace{\frac{1}{r}\cot\theta\frac{\partial A_\phi}{\partial r}}_{\text{II}}$$

$$-\underbrace{\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial A_r}{\partial\phi}}_{\text{I}} + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial^2 A_r}{\partial\theta\partial\phi}}_{\text{IX}} - \underbrace{\frac{1}{r^2}\frac{\partial A_\phi}{\partial\theta}}_{\text{IV}} - \underbrace{\frac{1}{r}\frac{\partial^2 A_\phi}{\partial\theta\partial r}}_{\text{VI}} + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\phi}}_{\text{III}} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial^2 A_\theta}{\partial\phi\partial r}}_{\text{V}}$$

$$-\underbrace{\frac{1}{r^2\sin\theta}\frac{\partial^2 A_r}{\partial\phi\partial\theta}}_{\text{IX}}$$

$$= 0$$

Problem 2.3

A

Cartesian	Cylindrical	Spherical
$h_x = 1$	$h_r = 1$	$h_r = 1$
$h_y = 1$	$h_\phi = r$	$h_\theta = r$
$h_z = 1$	$h_z = 1$	$h_\phi = r\sin\theta$

B

$$\begin{aligned} df &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw \\ &= \nabla f \cdot \overline{dl} \\ &= \nabla f \cdot \left[h_u du \overline{i_u} + h_v dv \overline{i_v} + h_w dw \overline{i_w}\right] \end{aligned}$$

$$(\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u}; \quad (\nabla f)_v = \frac{1}{h_v}\frac{\partial f}{\partial v}; \quad (\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w}$$

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\overline{i_u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\overline{i_v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\overline{i_w}$$

C

$$dS_u = h_v h_w dv dw; \quad dS_v = h_u h_w du dw; \quad dS_w = h_u h_v du dv$$

$$dV = h_u h_v h_w du dv dw$$

D

$$\begin{aligned} \Phi = \oint_S \overline{A}\cdot\overline{dS} = & \underbrace{\int_u A_u h_v h_w dv dw}_{1} - \underbrace{\int_{u-\Delta u} A_u h_v h_w dv dw}_{1'} \\ & + \underbrace{\int_{v+\Delta v} A_v h_u h_w du dw}_{2} - \underbrace{\int_v A_v h_u h_w du dw}_{2'} \\ & + \underbrace{\int_{w+\Delta w} A_w h_u h_v du dv}_{3} - \underbrace{\int_w A_w h_u h_v du dv}_{3'} \end{aligned}$$

$$= \left\{ \frac{A_u h_v h_w|_u - A_u h_v h_w|_{u-\Delta u}}{\Delta u} + \frac{A_v h_u h_w|_{v+\Delta v} - A_v h_u h_w|_v}{\Delta v} + \frac{A_w h_u h_v|_{w+\Delta w} - A_w h_u h_v|_w}{\Delta w} \right\} \Delta u \Delta v \Delta w$$

$$\nabla\cdot\overline{A} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0 \\ \Delta w \to 0}} \frac{\oint_S \overline{A}\cdot\overline{dS}}{\Delta V} = \frac{\oint_S \overline{A}\cdot\overline{dS}}{h_u h_v h_w \Delta u \Delta v \Delta w}$$

$$= \frac{1}{h_u h_v h_w}\left[\frac{\partial(h_v h_w A_u)}{\partial u} + \frac{\partial(h_u h_w A_v)}{\partial v} + \frac{\partial(h_u h_v A_w)}{\partial w}\right]$$

Curl

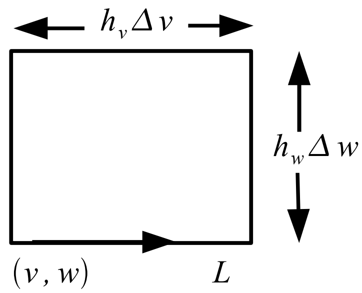

Figure 23: A diagram depicting how to calculate Curl for generalized right-handed orthogonal curvilinear coordinates (Image by MIT OpenCourseWare).

$$(\nabla \times \overline{A})_u = \lim_{\substack{\Delta v \to 0 \\ \Delta w \to 0}} \frac{\oint_L \overline{A} \cdot \overline{dl}}{h_v h_w \Delta v \Delta w}$$

$$\oint_L \overline{A} \cdot \overline{dl} = \left[A_v h_v \Delta v|_w - A_v h_v \Delta v|_{w+\Delta w}\right] + \left[A_w h_w \Delta w|_{v+\Delta v} - A_w h_w \Delta w|_v\right]$$

$$(\nabla \times \overline{A})_u = \lim_{\substack{\Delta v \to 0 \\ \Delta w \to 0}} \frac{1}{h_v h_w}\left\{\frac{\left[A_v h_v|_w - A_v h_v|_{w+\Delta w}\right]}{\Delta w} + \frac{\left[A_w h_w|_{v+\Delta v} - A_w h_w|_v\right]}{\Delta v}\right\}$$

$$= \frac{1}{h_v h_w}\left[\frac{\partial(h_w A_w)}{\partial v} - \frac{\partial(h_v A_v)}{\partial w}\right]$$

Similarly

$$(\nabla \times \overline{A})_v = \frac{1}{h_u h_w}\left[\frac{\partial(h_u A_u)}{\partial w} - \frac{\partial(h_w A_w)}{\partial u}\right]$$

$$(\nabla \times \overline{A})_w = \frac{1}{h_u h_v}\left[\frac{\partial(h_v A_v)}{\partial u} - \frac{\partial(h_u A_u)}{\partial v}\right]$$

E

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right]$$

Problem 2.4

Demo 4.7.1: Charge Induced in Ground Plane by Overhead Conductor

- This problem is analogous to high voltage power line over earth problem.
- We can use image charge assumption to determine the induced charge on ground plane.
- We make use of a plane probe, insulated from the ground plane, to measure the charge induced on it by the cylindrical conductor.
- The surface charge density distribution is proportional to the voltage applied to the cylinder.
- Because the probe voltage is time derivative of the applied voltage, the probe signal is 90° out-of-phase.
- As the probe is moved out from below the conductor cylinder, charge induced on its surface decreases.
- Electric field at low frequencies do not penetrate conducting body.

Demo 10.2.1: Edgerton's Boomer

- Illustrates induction of a current in a conductor, subjected to a time varying magnetic field.
- Illustrates the interplay of laws of Faraday, Ampere, and Ohm.
- MQS conditions apply.
- 4kV capacitor voltage.
- Use of a coil probe to determine voltage induced by time varying magnetic field.
- Magnetic field produced by the coil is non-uniform, similar to that of a dipole.
- Prediction of an arc due to high electric field intensity between almost touching wires.
- Can be used to chape materials.
- Or shoot up metal plates.

Problem 3.1

Demo 8.4.1: Surface used to define flux linkage

- Copper wires were wound on a circular wooden rod, and coil inductances are measured.
- If number of turns is doubled, the measured inductance value is quadrupled
- When number of coils is doubled as well as the length, the inductance value is doubled
- Results agree well with the theoretical equation: $L = \frac{\mu_0 A N^2}{d}$

Demo 8.2.1: Field of circular cylindrical solenoid

- Magnetic field of a large cylindrical solenoid is measured using magnetometer probe and transverse probe
- Field insude is uniform and in the axial direction, the field outside the solenoid is close to 0.

- Circular cylindrical solenoid is analogous to plane parallel capacitors for having uniform B-field inside and zero B-field outside
- A transverse probe (which measures B-field intensity transverse to its surface) is used to determine that the B-field inside is axial.
- Through a slit cut on the cylindrical solenoid, the transverse probe is used to observe the discontinuity of the magnetic field intensity between inside and outside.

Demo 8.2.2: Field of square pair of coils

- 2 square coils 45cm apart, each has 50 turns and size of 45cm length.
- Axial magnetometer probe is used to measure the intensity of axial magnetic field
- Theoretical curve is well matched (see Figure 24)

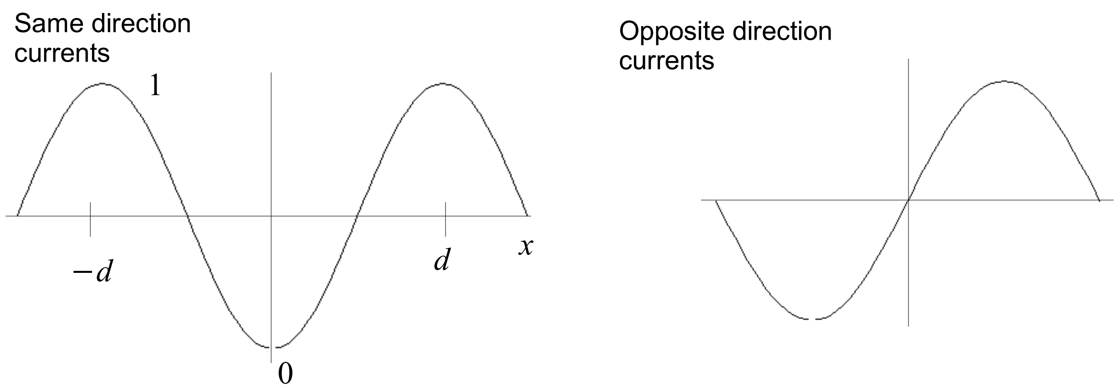

Figure 24: Two graphs showing current along the x-axis of the coil with same direction current and opposite direction current (Image by MIT OpenCourseWare).

Problem 4.1

A

Linear media $\Rightarrow \overline{J} = \sigma \overline{E}$, by symmetry $\overline{E}$ is only in x direction, thus current density $\overline{J}$ is also in x direction

$$\overline{E} = E_x \overline{i_x} \;\; , \;\; \overline{J} = J_x \overline{i_x} \quad \sigma = \sigma(x) \; - \text{ a function of x}$$

Conservation of charge $\nabla \cdot \overline{J_f} + \frac{\partial \rho_f}{\partial t} = 0$
In DC steady state $\dfrac{\partial \rho_f}{\partial \epsilon} = 0$

$$\Rightarrow \nabla \cdot \overline{J_f} = 0 \Rightarrow \frac{d}{dx} J_x = 0 \Rightarrow J_x = J_0 = \text{constant}$$
$$\Rightarrow \overline{J} = J_0 \overline{i_x}$$

For $\sigma(x) = \dfrac{\sigma_0}{1 + \frac{x}{s}}$

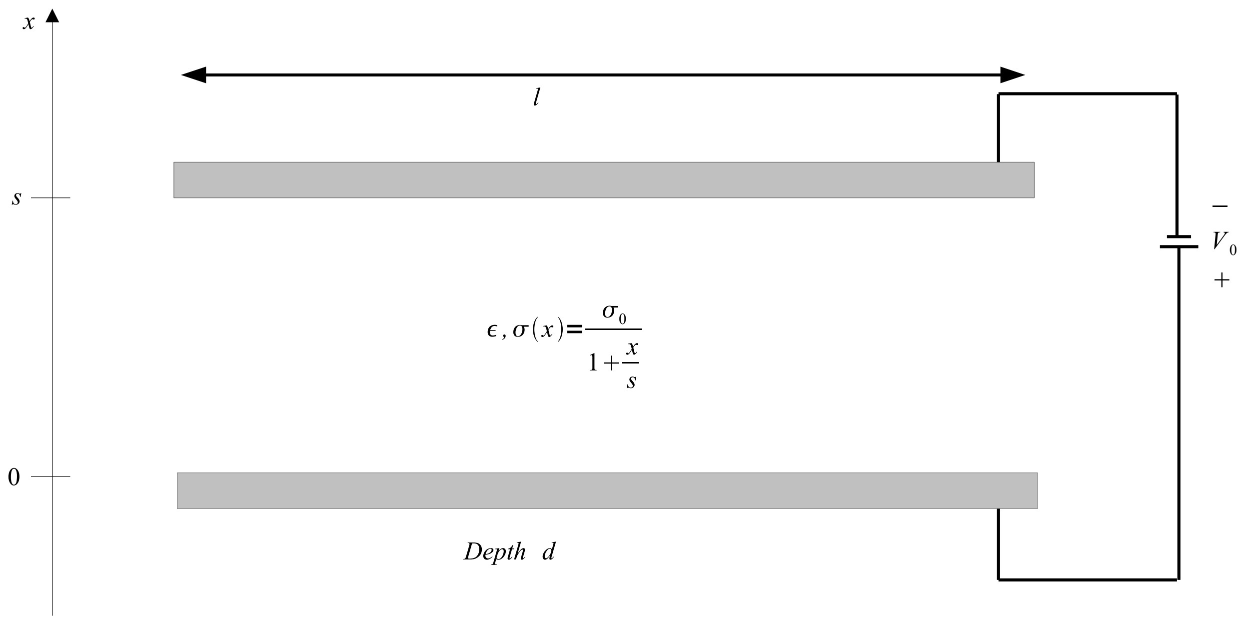

Figure 25: A diagram of parallel plate electrodes enclosing a lossy dielectric with x dependent conductivity (Image by MIT OpenCourseWare).

$$\overline{J} = \sigma\overline{E} \Rightarrow J_x = \sigma(x)E_x \Rightarrow E_x = \frac{J_x}{\sigma(x)} = \frac{J_0}{\frac{\sigma_0}{1+\frac{x}{s}}} = \frac{J_0}{\sigma_0}\left(1+\frac{x}{s}\right)$$

$$\int_0^s E_x dx = V_0 = \int_0^s \frac{J_0}{\sigma_0}\left(1+\frac{x}{s}\right)dx = \frac{J_0}{\sigma_0}\left(x+\frac{x^2}{2s}\right)\Bigg|_0^s = \frac{J_0}{\sigma_0}\left(s+\frac{s}{2}\right) = \frac{3}{2}\frac{J_0 s}{\sigma_0}$$

$$\Rightarrow \boxed{J_0 = \tfrac{2\sigma_0 V_0}{3s}}$$

$$\Rightarrow E_x(x) = \frac{2\cancel{\sigma_0}V_0}{3s\cancel{\sigma_0}}\left(1+\frac{x}{s}\right) = \boxed{\tfrac{2V_0}{3s}\left(1+\tfrac{x}{s}\right)}$$

Total current $\Rightarrow I = J_0(\text{Area}) = J_0 ld = \frac{2\sigma_0 V_0 fd}{3s}$

$$\Rightarrow \text{Resistance} = \frac{V_0}{I} = \boxed{\frac{3s}{2\sigma_0 ld}}$$

B

$$\nabla\cdot\overline{D} = \rho_f \Rightarrow \rho_f = \epsilon\frac{dE_x}{dx} = \epsilon\frac{d}{dx}\left(\frac{2V_0}{3s}\left(1+\frac{x}{s}\right)\right)$$

$$\boxed{\rho_f = \tfrac{\epsilon 2V_0}{3s^2}}$$

Boundary conditions at $x = 0$ and $x = s$ will provide the surface charges densities (see Figure 26).

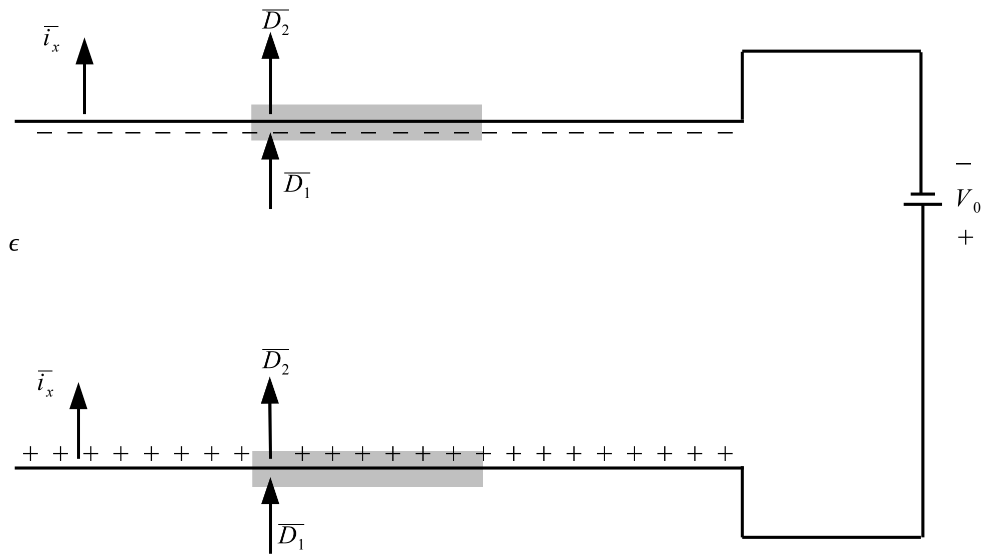

Figure 26: A diagram showing the parallel plate electrodes with displacement field vectors showing the sources of the surface charge densities (Image by MIT OpenCourseWare).

$$\overline{n} \cdot (\overline{D_2} - \overline{D_1}) = \sigma_{sf}$$

$$\Rightarrow \text{ at } x = 0 \quad \sigma_{sf} = \epsilon E_x|_{x=0} = \frac{2\epsilon V_0}{3s}$$

$$\text{at } x = 0 \quad \sigma_{sf} = -\epsilon E_x|_{x=s} = -\frac{4\epsilon V_0}{3s}$$

C

$$q_{\text{total volume}} = ld \int_0^s \rho_f dx = ld \frac{2\epsilon V_0}{3s^2} s = \boxed{\frac{2\epsilon V_0 ld}{3s}}$$

$$\left.\begin{array}{l} \text{at } x = 0 \ q_{\text{surface}} = ld\sigma_s f|_{x=0} = \frac{2\epsilon V_0 \rho d}{3s} \\ \text{at } x = s \ \ q_{\text{surface}} = ld\sigma_{sf}|_{x=s} = -\frac{4\epsilon V_0 ld}{3s} \end{array}\right\} \Rightarrow q_{\text{total surface}} = -\frac{2\epsilon V_0 ld}{3s} = -q_{\text{total volume}}$$

$$q_{\text{surface}}(x = 0) + q_{\text{surface}}(x = s) + q_{\text{total volume}} = 0$$

Problem 4.2

A

$\rho_f = 0$ inside the dielectric media $\Rightarrow$ use Gauss' Laws for $(a < r < b)$

$\Rightarrow \nabla \cdot \overline{D} = 0$ (due to symmetry only radial $\overline{D}$ component exists)

$$\Rightarrow \nabla \cdot \overline{D} = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) = 0 \Rightarrow r^2 D_r = c \Rightarrow \text{ constant} \Rightarrow \boxed{D_r = \frac{c}{r^2}}$$

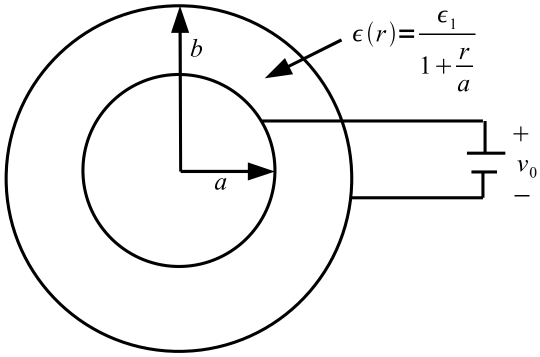

Figure 27: Concentric spherical electrodes enclose a dielectric with permittivity that varies with r with no volume charge in the dielectric (Image by MIT OpenCourseWare).

$$\Rightarrow \epsilon(r)E_r = D_r \Rightarrow E_r = \frac{D_r}{\epsilon(r)} = \frac{c}{r^2\epsilon_1}\left(1+\frac{r}{a}\right) \Rightarrow \boxed{E_r = \frac{c}{\epsilon_1}\left(\frac{1}{r^2}+\frac{1}{ra}\right)}$$

$$V_0 = \int_{r=a}^{b} E_r dr = \int_{r=a}^{b} \frac{c}{\epsilon_1}\left(r^{-2}+\frac{1}{a}r^{-1}\right)dr = \frac{c}{\epsilon_1}\left(-\frac{1}{r}+\frac{1}{a}\ln r\right)\Big|_a^b$$

$$= \frac{c}{\epsilon_1}\left(-\frac{1}{b}+\frac{1}{a}\ln b+\frac{1}{a}-\frac{1}{a}\ln a\right)$$

$$V_0 = \frac{c}{\epsilon_1}\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}\right)$$

$$\Rightarrow \boxed{c = \frac{V_0\epsilon_1}{\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}}}$$

$$\Rightarrow \boxed{\overline{E} = \frac{V_0}{\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}}\left(\frac{1}{r^2}+\frac{1}{ra}\right)\bar{i}_r} \text{ for } a<r<b$$

$$\overline{E} = -\nabla\Phi \Rightarrow E_r = -\frac{\partial\Phi}{\partial r} = \frac{V_0}{\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}}\left(\frac{1}{r^2}+\frac{1}{a}\frac{1}{r}\right)$$

$$\Rightarrow \Phi = -\frac{V_0}{\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}}\int\left(\frac{1}{r^2}+\frac{1}{a}\frac{1}{r}\right)dr$$

$$= -\frac{V_0}{\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}}\left(-\frac{1}{r}+\frac{1}{a}\ln r+\underbrace{d}_{\text{constant}}\right)$$

$$\Phi|_{r=a} = V_0 = -\frac{V_0}{\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}}\left(-\frac{1}{a}+\frac{1}{a}\ln a+d\right)$$

$$\cancel{\frac{1}{a}}-\frac{1}{b}+\frac{1}{a}\ln b-\cancel{\frac{1}{a}\ln a} = \cancel{\frac{1}{a}}-\cancel{\frac{1}{a}\ln a}-d$$

$$\boxed{d = \frac{1}{b}-\frac{1}{a}\ln b} \text{ or use } \Phi|_{r=b}=0 \text{ to get d}$$

$$\Rightarrow \boxed{\Phi(r) = -\frac{V_0}{\frac{1}{a}-\frac{1}{b}+\frac{1}{a}\ln\frac{b}{a}}\left(-\frac{1}{r}+\frac{1}{b}+\frac{1}{a}\ln\frac{r}{b}\right)} \text{ for } a<r<b$$

B

Using B.C. at $r = a$ and at $r = b$: $\overline{n} \cdot (\overline{D_2} - \overline{D_1}) = \sigma_{SR}$

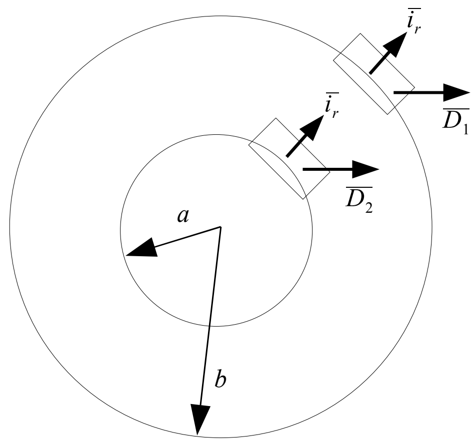

Figure 28: The surface charge at $r = a$ is $D_{2r}(r = a)$ and at $r = b$ is $-D_{1r}(r = b)$ (Image by MIT OpenCourseWare).

$$\text{at } r = a \quad \sigma_{sf} = D_r|_{r=a} = \frac{1}{a^2}\frac{V_0\epsilon_1}{\frac{1}{a} - \frac{1}{b} + \frac{1}{a}\ln\frac{b}{a}} = \frac{V_0\epsilon_1}{a - \frac{a^2}{b} + a\ln ba}$$

$$\text{at } r = b \quad \sigma_{sf} = -D_r|_{r=b} = -\frac{1}{b^2}\frac{V_0\epsilon_1}{\frac{1}{a} - \frac{1}{b} + \frac{1}{a}\ln\frac{b}{a}} = \frac{-V_0\epsilon_1}{\frac{b^2}{a} - b + \frac{b^2}{a}\ln\frac{b}{a}}$$

C

Total charge on inner electrode

$$q = \sigma_s|_{r=a} 4\pi a^2 = \frac{4\pi a^2 V_0 \epsilon_1}{a - \frac{a^2}{b} + a\ln\frac{b}{a}} \Rightarrow \boxed{\text{capacitance } = \frac{q}{V} = \frac{4\pi\epsilon_1}{\frac{1}{a} - \frac{1}{b} + \frac{1}{a}\ln\frac{b}{a}}}$$

Problem 4.3

$$\rho_f(t=0) = \begin{cases} \dfrac{\rho_0 r^2}{a_0^2} & 0 < r < a_0 \\ 0 & r > a_0 \end{cases}$$

<u>Charge relaxation</u>

$$\left.\begin{array}{l} \text{Conservation of charge: } \nabla \cdot \overline{J}_f + \dfrac{\partial \rho_f}{\partial t} = 0 \Rightarrow \dfrac{\sigma}{\epsilon}\rho_f + \dfrac{\partial \rho_f}{\partial t} = 0 \\ \left.\begin{array}{l} \text{Gauss' Law } \nabla \cdot \overline{E} = \dfrac{\rho_f}{\epsilon} \\ \text{Linear Media } \overline{J_f} = \sigma \overline{E} \end{array}\right\} \nabla \cdot \overline{J}_f = \sigma \nabla \cdot \overline{E} = \sigma \dfrac{\rho_f}{\epsilon} \end{array}\right\} \Rightarrow \boxed{\rho_f(t) = \rho_f(t=0)e^{-t/\tau}, \quad \tau = \frac{\epsilon}{\sigma}}$$

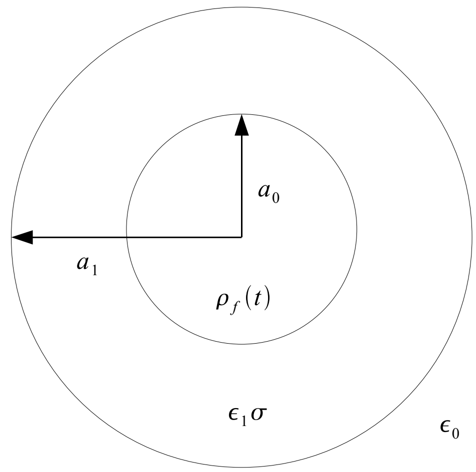

Figure 29: An infinitely long cylinder of radius a_0 with initial charge density $\rho_f(t=0) = \dfrac{\rho_0 r^2}{a_0^2}$ for $r < a_0$ and zero for $r > a_0$ (Image by MIT OpenCourseWare).

$$\Rightarrow \rho_f(r,t) = \begin{cases} \dfrac{\rho_0 r^2}{a_0^2} e^{-t/\tau} & ;\ 0 < r < a_0 \\ 0 & ; r > a_0 \end{cases}$$

Using Gauss' Law

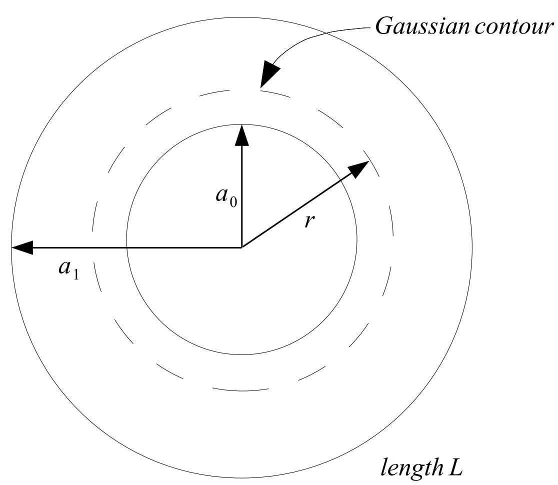

Figure 30: An infinitely long cylinder showing a Gaussian Contour for $a_0 < r < a_1$ (Image by MIT OpenCourseWare).

$$\oint_S \epsilon \overline{E} \cdot d\overline{a} = \int_V \rho_f dV$$

$$2\pi r L E_r \epsilon = \text{charge enclosed}$$

$$\cancel{2\pi} r \cancel{L} E_r \epsilon = \cancel{2\pi} \int_0^r \rho_f(r',t) r' dr' \Rightarrow E_r = \frac{1}{r\epsilon} \underbrace{\int_0^r \rho_f(r',t) r' dr'}_{\text{charge enclosed}}$$

$$\overline{E}_r = \overline{i}_r \begin{cases} \frac{1}{r\epsilon} \int_0^r \frac{\rho_0 (r')^2}{a_0^2} e^{-t/\tau} r' dr' = \frac{1}{r\epsilon} \left(\frac{\rho_0}{a_0^2} e^{-t/\tau} \frac{r^4}{4} \right) = \frac{\rho_0 r^3}{4 a_0^2 \epsilon} e^{-t/\tau} & ; 0 < r < a_0 \\ \frac{1}{r\epsilon} \int_0^{a_0} \frac{\rho_0 (r')^2}{a_0^2} e^{-t/\tau} r' dr' = \frac{1}{r\epsilon} \left(\frac{\rho_0}{a_0^2} e^{-t/\tau} \frac{a_0^4}{4} \right) = \frac{\rho_0 a_0^2}{4 \epsilon r} e^{-t/\tau} & ; a_0 < r < a_1 \\ \frac{1}{r\epsilon_0} \int_0^{a_0} \frac{\rho_0 (r')^2}{a_0^2} r' dr' = \frac{1}{r\epsilon_0} \left(\frac{\rho_0}{a_0^2} \frac{a_0^4}{4} \right) = \frac{\rho_0 a_0^2}{4 \epsilon_0 r} & ; r > a_1 \end{cases}$$

Using B.C., to find surface charge at $r = a$,

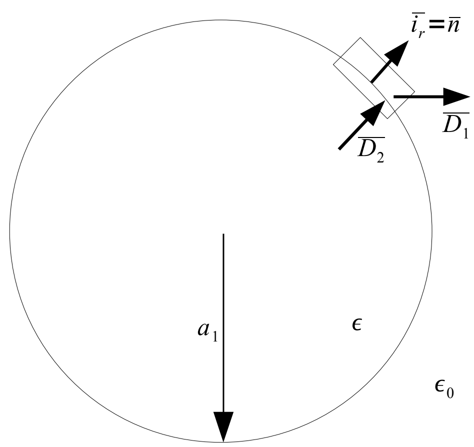

Figure 31: An infinitely long cylinder showing vectors $\overline{D}_1$ and $\overline{D}_2$ that determine the surface charge density σ_{sf} (Image by MIT OpenCourseWare).

$$\overline{n} \cdot (\overline{D}_2 - \overline{D}_1) = \sigma_{sf}$$

$$\sigma_{sf}|_{r=a_1} = \epsilon_0 E_r|_{r=a_1^+} - \epsilon E_r|_{r=a_1^-} = \epsilon_0 \frac{\rho_0 a_0^2}{4 a_1 \epsilon_0} - \epsilon \frac{\rho_0 a_0^2}{4 a_1 \epsilon} e^{-t/\tau}$$

$$\Rightarrow \boxed{\sigma_{sf} = \frac{\rho_0 a_0^2}{4 a_1} \left(1 - e^{t/\tau} \right)}$$

Problem 4.4

Goal: Place an image charge (or image charges) inside the cylinder such that cylinder remains an equipotential surface. This simplifies the problem to that of infinitely long line charges.

Figure 33 shows the $\overline{E}$ field due to a line charge λ

$$\epsilon_0 \oint_S \overline{E} \cdot d\overline{a} = \int_V \rho dv$$

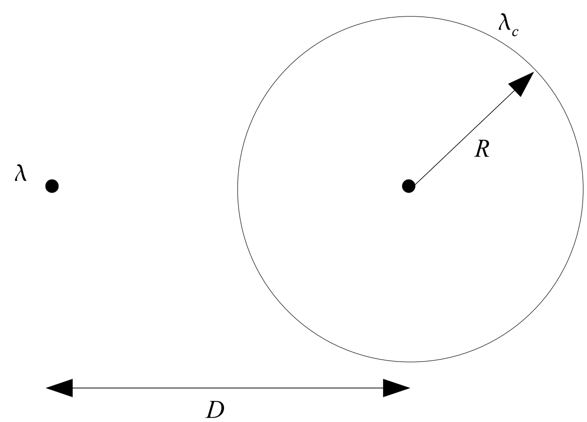

Figure 32: A diagram of an infinitely long line charge λ a distance D from the center of an infinitely long cylinder of radius R with charge per unit length λ_c (Image by MIT OpenCourseWare).

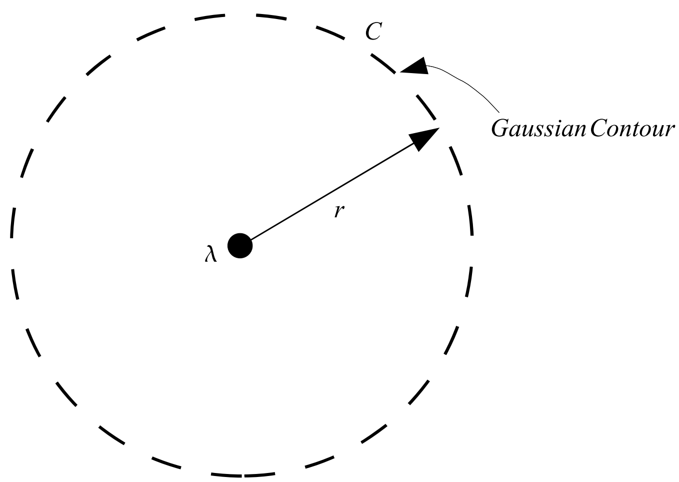

Figure 33: A diagram of the Gaussian Contour at radius r around an infinitely long line charge (Image by MIT OpenCourseWare).

$$\epsilon_0 E_r 2\pi r \not{l} = \lambda \not{l}$$

$$\boxed{E_r = \frac{\lambda}{2\pi\epsilon_0 r}}$$

Potential due to a line charge:

$$\overline{E} = -\nabla\Phi \Rightarrow E_r = -\frac{d}{dr}\Phi = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$\Rightarrow \boxed{\Phi = -\frac{\lambda}{2\pi\epsilon_0}\ln r + \text{ constant}}$$

Equipotential surfaces due to two line charges of magnitude λ and $-\lambda$ are cylinders. See Figure 2.24 in Zahn's book.

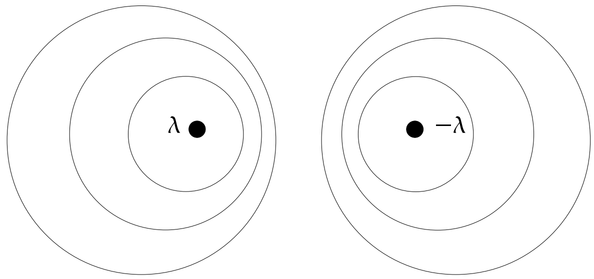

Figure 34: A pair of opposite polarity line charges have circular cylindrical equipotential surfaces (Image by MIT OpenCourseWare).

Thus, if an image charge $-\lambda$ is placed inside the cylinder at $\frac{R^2}{D}$ (see Zahn p. 98) distance from the center, the condition for an equipotential surface at the cylinder will be satisfied.

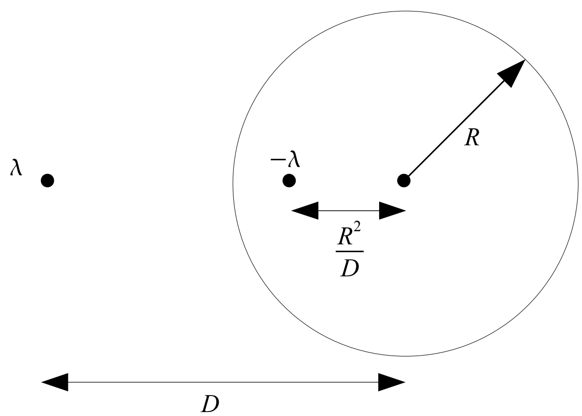

Figure 35: A diagram showing an image line charge $-\lambda$ a distance R^2/D away from the center of the cylinder and another line charge λ a distance D away from the center. (Image by MIT OpenCourseWare).

In order to satisify the condition that the surface of the cylinder holds a charge per unit length λ_c, and maintaining that the surface is equipotential, another image charge must be placed at the center of the cylinder. This image charge has value $\lambda + \lambda_c$ to keep the total cylinder charge at λ_c.

Problem reduces to (see figure 36):
$\Rightarrow$ Force on cylinder is due to the force on the two image line charges $-\lambda$ and $\lambda + \lambda_c$

$$\Rightarrow \boxed{f_x = \frac{\lambda}{2\pi\epsilon_0}\left[\frac{-\lambda}{D - \frac{R^2}{D}} + \frac{(\lambda + \lambda_c)}{D}\right]}$$

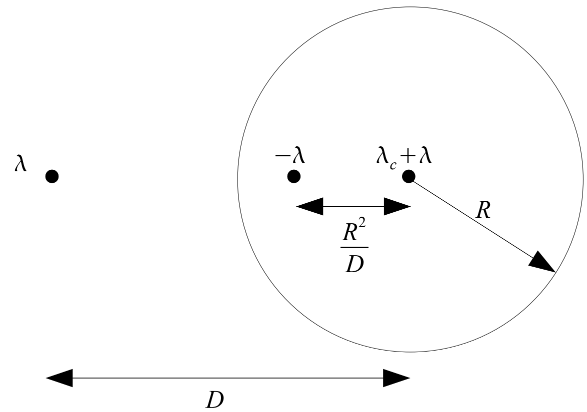

Figure 36: A diagram showing the locations of all the line charges and cylinder giving the total line charge on the cylinder as λ_c (Image by MIT OpenCourseWare).

Problem 4.5

1.4.1 Magnetic field of line current

- Ampere's law predicts B-field intensity is inversely proportional to radial distance r.
- Hall effect probe is used to measure B-field intensity.
- Hall effect proble measures B-field intensity perpendicular to its flat surface.
- Magnetic field due to a wire is non-zero only in ϕ direction.

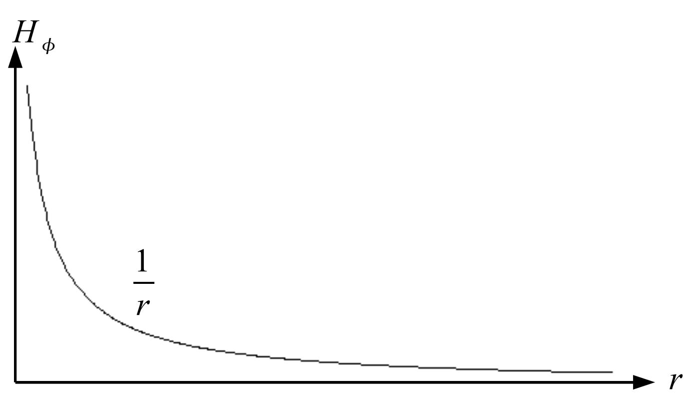

Figure 37: A graph showing the B-field radial dependence of $1/r$ for a long line current as predicted by Ampere's Law (Image by MIT OpenCourseWare).

1.6.1 Voltmeter Reading Induced by Magnetic Induction

- Contour C is given by the coil shown in Figure 38
- Use the coil shown in Figure 38 to find the magnetic field associated for a current carrying wire.

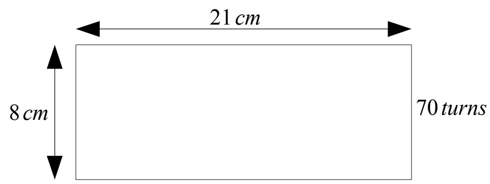

Figure 38: A diagram showing Contour C (Image by MIT OpenCourseWare).

- Faraday integral law shows how a voltmeter reading induced by magnetic induction provides a measurement of magnetic flux density.
- Observe a change in phase as coil is moved from below the wire to above the wire.
- Coil voltage is 90° out of phase with wire current.

6.6.1 An Artificial Dielectric

- Artificial dielectric is constructed of an array of conducting spheres (ping-pong balls with conductive coating). See Figure 39

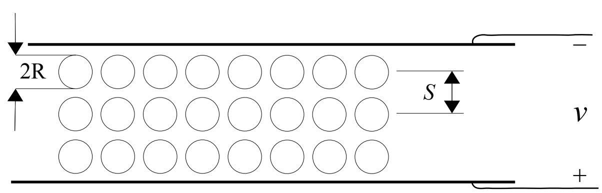

Figure 39: A diagram showing an artificial dielectric constructed of an array of conducting spheres (Image by MIT OpenCourseWare).

- Application of voltage v to the electrodes results in the spheres acquiring negative and positive charges on their poles.
- Insertion of dielectric array between the plane parallel conductors increases the capacitance.

9.4.1 Measurement of B-H Characteristic

- Magnetizable material
- Polycrystalline and ferromagnetic materials at domain level have randomly oriented magnetic moments that tend to cancel in the absence of applied field.
- Those domains align with the applied magnetic field.
- However phase delay develops between magnetization and applied field resulting in power dissipation.
- Hysteresis loop

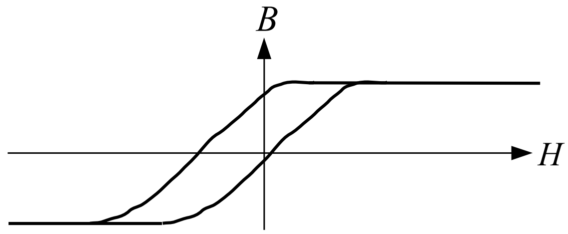

Figure 40: A magnetization hysteresis loop (Image by MIT OpenCourseWare).

Problem 5.1

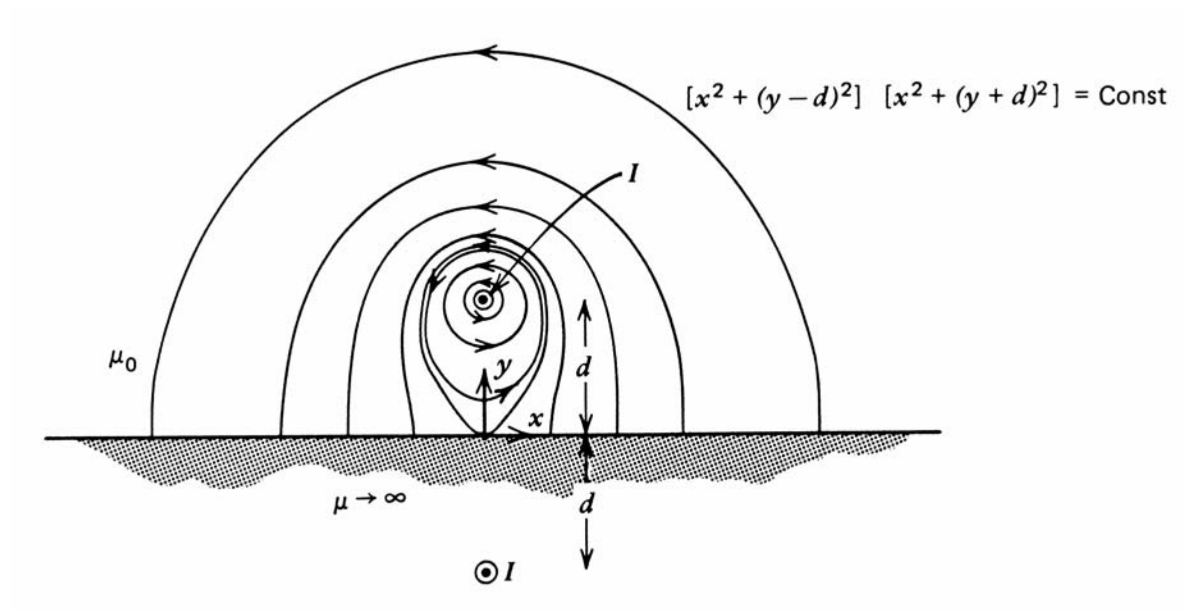

Figure 41: A diagram showing the magnetic field lines from a line current I of infinite extent in free space above a plane of material of infinite magnetic permeability.

Line current I of infinite extent above a plane of material of infinite permeability, $\mu \to \infty$.

A

$\vec{B} = \mu\vec{H} \Rightarrow$ for $\mu \to \infty$, in order to have B finite, we need H zero $\Rightarrow$ continuity of normal $\vec{B}$ and tangential $\vec{H}$ at the surface.

B

Using method of images to satisify boundary conditions at $y = 0$ for medium where $\mu \to \infty$ requires $\Rightarrow H_x = H_z = 0$ at $y = 0$

For a line current at origin (see Figure 43)

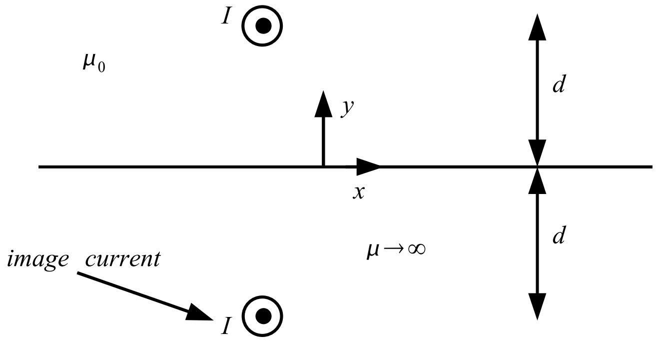

Figure 42: A diagram showing how to apply the method of images for a line current I in free space above an infinite magnetic permeability material ($\mu \to \infty$)(Image by MIT OpenCourseWare).

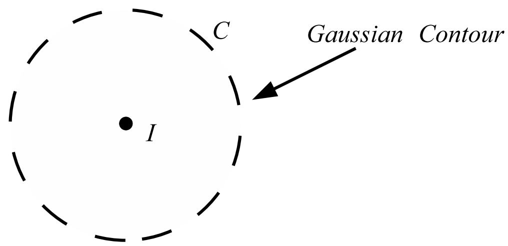

Figure 43: A diagram depicting a Gaussian Contour to determine the magnetic field from an infinitely long line current I (Image by MIT OpenCourseWare).

$$\oint_C \vec{H} \cdot d\vec{l} = I \Rightarrow H_\phi = \frac{I}{2\pi r}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\vec{i_\phi} \quad \text{, since} \quad \vec{B} = \nabla \times \vec{A} \Rightarrow -\frac{\partial A_z}{\partial r} = \frac{I\mu_0}{2\pi r}$$

$$\Rightarrow A_z = -\frac{I\mu_0}{2\pi}\ln r + \text{ constant}$$

$$\Rightarrow \text{ for line currents } I \text{ at } z = d \text{ and } I \text{ at } z = -d$$

$$\boxed{A_z = -\frac{I\mu_0}{2\pi}\left\{\ln\left[\sqrt{x^2 + (y-d)^2}\right] + \ln\left[\sqrt{x^2 + (y+d)^2}\right]\right\}}$$

$$\Rightarrow \boxed{A_z = -\frac{I\mu_0}{4\pi}\ln\left\{\left[x^2 + (y-d)^2\right]\left[x^2 + (y+d)^2\right]\right\}}$$

C

$$\vec{B} = \nabla \times \vec{A} = -\frac{\partial}{\partial x}A_z\vec{i_y} + \frac{\partial}{\partial y}A_z\vec{i_x}$$

$$= \frac{I\mu_0}{4\pi}\frac{2x\left[x^2 + (y+d)^2\right] + 2x\left[x^2 + (y-d)^2\right]}{\left[x^2 + (y-d)^2\right]\left[x^2 + (y+d)^2\right]}\vec{i_y} - \frac{I\mu_0}{4\pi}\frac{2(y-d)\left[x^2 + (y+d)^2\right] + 2(y+d)\left[x^2 + (y-d)^2\right]}{\left[x^2 + (y-d)^2\right]\left[x^2 + (y+d)^2\right]}\vec{i_x}$$

$$\boxed{\vec{B} = -\frac{I\mu_0}{2\pi}\left[\frac{(y-d)\vec{i_x} - x\vec{i_y}}{x^2+(y-d)^2} + \frac{(y+d)\vec{i_x} - x\vec{i_y}}{x^2+(y+d)^2}\right]}$$

D

Force is applied on the line current due to the image line current

Force per unit length:

$$\vec{F} = \vec{I} \times \vec{B} \leftarrow \text{field due to image charge at } (x=0, y=-d)$$

$$= \vec{I}\vec{i_z} \times \left(\frac{\mu_0 I}{2\pi}\right)\frac{1}{2d}\vec{i_x}$$

$$= \boxed{-\frac{\mu_0 I^2}{4\pi d}\vec{i_y}} \text{ so line current is attracted to the surface}$$