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6.641 Electromagnetic Fields, Forces, and Motion
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Problem Set 7 - Solutions

Problem 7.1

The electric field intensity between the plates is

$$E = v/a$$

Hence, the surface charge adjacent to the free space region on the upper plate is

$$\sigma_f = \varepsilon_0 v/a$$

while that next to the nonlinear dielectric slab is

$$\sigma_f = \alpha \frac{v^3}{a^3} + \varepsilon_0 \frac{v}{a}$$

It follows that the total charge on the upper plate is

$$q = \frac{dx\varepsilon_0 v}{a} + d(l-x) \left(\frac{\alpha v^3}{a^3} + \frac{\varepsilon_0 v}{a} \right)$$

The electric co-energy is

$$W'_e = \int q dv = \frac{dl\varepsilon_0 v^2}{2a} + \frac{d(l-x)\alpha v^4}{4a^3}$$

Then, the force of electrical origin is

$$f^e = \left. \frac{\partial W'_e}{\partial x} \right|_{v \text{ constant}} = -\frac{d\alpha v^4}{4a^3}$$

Problem 7.2

We ignore fringing fields. Then the electric field is completely between the center plate and the outer plates, where it has the value $E = v/b$. The constraints on the electrical terminals further require that $v = V_0 - Ax$. The surface charge on the outer plates is $\varepsilon_0 \frac{v}{b}$ and hence the total charge q on these plates is

$$q = 2(a-x) \frac{d\varepsilon_0}{b} v \tag{1}$$

It follows that the co-energy is

$$W'_e = (a-x) \frac{d\varepsilon_0}{b} v^2 \tag{2}$$

and the electrical force is

$$f^e = \left. \frac{\partial W'_e}{\partial x} \right|_{v \text{ constant}} = -\frac{d\varepsilon_0}{b} v^2 \tag{3}$$

Finally, we use the electrical circuit conditions to write

$$f^e = -\frac{d\varepsilon_0}{b} (V_0 - Ax)^2 \quad (4)$$

The major point to be made in this situation is this. One might substitute the voltage, as it depends on x , into (2) before taking the derivative. This clearly gives an answer not in agreement with (4). We have assumed in writing (3) that the variables (v, x) remain thermodynamically independent until after the force has been found. Of course, in the actual situation, external constraints relate these variables, but these constraints can only be introduced with care in the energy functions. To be safe they should not be introduced until after the force has been found.

Problem 7.3

A

The magnetic field intensity in the gap must first be related to the excitation current. From Ampere's law

$$Ni = dH_d + xH_x \quad (5)$$

where the fields H_d and H_x are directed counterclockwise around the magnetic circuit when they are positive. These fields are further related because the magnetic flux into the movable member must equal that out of it

$$\mu_0 wbH_d = \mu_0 waH_x \quad (6)$$

From these two expressions

$$H_x = \frac{Ni}{\left(\frac{da}{b} + x\right)} \quad (7)$$

The flux linked by the electrical terminals is $\lambda = N\mu_0 awH_x$, which in view of (7) is

$$\lambda = Li; L = \frac{N^2\mu_0 aw}{\left(\frac{da}{b} + x\right)} \quad (8)$$

B

The system is electrically linear. Hence, $W_m = \frac{1}{2}\lambda^2/L$ (See sec. 3.1.2b) and from (8),

$$W_m = \frac{1}{2}\lambda^2 \frac{\left(\frac{da}{b} + x\right)}{N^2\mu_0 aw} \quad (9)$$

C

From conservation of energy $f^e = -\frac{\partial W_m}{\partial x} \Big|_{\lambda \text{ constant}}$, $W_m = W_m(\lambda, x)$. Hence,

$$f^e = -\frac{1}{2} \frac{\lambda^2}{(N^2\mu_0 aw)} \quad (10)$$

D

In view of (8) the current node equation can be written as (remember that the terminal voltage is $\frac{d\lambda}{dt}$)

$$I(t) = \frac{1}{R} \frac{d\lambda}{dt} + \frac{\lambda \left(\frac{da}{b} + x\right)}{N^2\mu_0 aw} \quad (11)$$

E

The inertial force due to the mass M must be equal to two other forces, one due to gravity and the other f^e . Hence,

$$M \frac{d^2x}{dt^2} = Mg - \frac{1}{2} \frac{\lambda^2}{N^2 \mu_0 a w} \quad (12)$$

(11) and (12) are the required equations of motion, where (λ, x) are the dependent variables.

Problem 7.4**A**

From the terminal relations, the electrical co-energy is (Table 3.1.1)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2$$

or

$$W'_m = \frac{1}{2} a x_1^2 i_1^4 + b x_2^2 x_1 i_1 i_2 + \frac{1}{4} c x_2^2 i_2^4$$

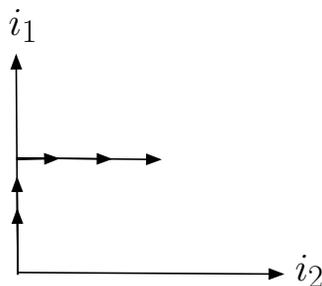


Figure 1: Co-energy computed by integration in the $i_1 - i_2$ plane (Image by MIT OpenCourseWare.)

B

It follows that the required forces are

$$f_1^e = \left. \frac{\partial W'_m}{\partial x_1} \right|_{i_1, i_2, x_2 \text{ constant}} = \frac{1}{2} a x_1 i_1^4 + b x_2^2 i_1 i_2$$

$$f_2^e = \left. \frac{\partial W'_m}{\partial x_2} \right|_{i_1, i_2, x_1 \text{ constant}} = 2b x_2 x_1 i_1 i_2 + \frac{1}{2} c x_2 i_2^4$$

C

There are four equations of motion in the dependent variables i_1, i_2, x_1 and x_2 ; two of these are the electrical voltage equations, which in view of the terminal equations for the λ 's are

$$-i_1 R_1 = \frac{d}{dt}(ax_1^2 i_1^3 + bx_2^2 x_1 i_2)$$

$$v_2(t) - i_2 R_2 = \frac{d}{dt}(bx_2^2 x_1 i_1 + cx_2^2 i_2^3)$$

and two are the mechanical force equations

$$0 = \frac{1}{2}ax_1 i_1^4 + bx_2^2 i_1 i_2 - Kx_1$$

$$0 = 2bx_2 x_1 i_1 i_2 + \frac{1}{2}cx_2 i_2^4 - B \frac{dx_2}{dt}$$

Problem 7.5**A**

From the terminal pair relation, the coenergy is given by

$$W'_m(i_1, i_2, \theta) = \frac{1}{2}(L_0 + M \cos 2\theta)i_1^2 + \frac{1}{2}(L_0 - M \cos 2\theta)i_2^2 + M \sin 2\theta i_1 i_2$$

so that the torque of electrical origin is

$$T^e = M(\sin 2\theta(i_2^2 - i_1^2) + 2 \cos 2\theta i_1 i_2)$$

B

For the two phase currents, as given,

$$i_2^2 - i_1^2 = -I^2 \cos 2\omega_s t$$

$$i_1 i_2 = I^2 \frac{1}{2} \sin 2\omega_s t$$

so that the torque T^e becomes

$$T^e = MI^2(-\sin 2\theta \cos 2\omega_s t + \sin 2\omega_s t \cos 2\theta)$$

or

$$T^e = MI^2 \sin(2\omega_s t - 2\theta)$$

Substitution of $\theta = \omega_m t + \gamma$ obtains

$$T^e = -MI^2 \sin(2(\omega_m - \omega_s)t + 2\gamma)$$

and for this torque to be constant, we must have the frequency condition

$$\omega_m = \omega_s$$

under which condition, the torque can be written as

$$T^e = -MI^2 \sin 2\gamma$$

C

To determine the possible equilibrium angles δ_0 , the perturbations and time derivatives are set to zero in the mechanical equations of motion.

$$T_0 = MI^2 \sin 2\gamma_0 \tag{13}$$

Here, we have written the time dependence in a form that is convenient if $\cos 2\gamma_0 > 0$, as it is at the points marked (s) in Fig. 2. Hence, these points are stable. At the points marked (u), the argument of the sin function and the denominator are imaginary, and the response takes the form of a sinh function. Hence, the equilibrium points indicated by (u) are unstable.

Graphical solutions of this expression are shown in Fig. 2. For there to be equilibrium values of γ the currents must be large enough that the torque can be maintained with the rotor in synchronism with the rotating field ($MI^2 > T_0$). Returning to the perturbation part of the equation of motion with $\omega_m = \omega_s$,

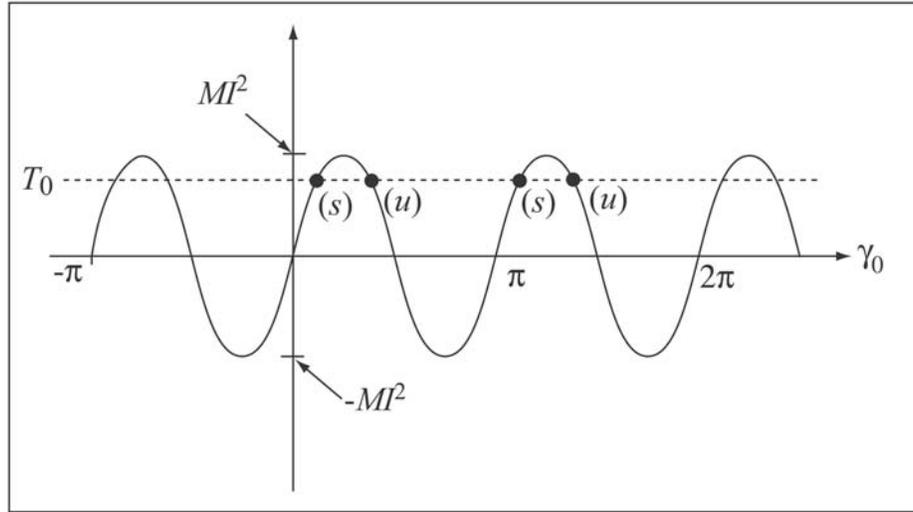


Figure 2: Graphical solutions for torque angle γ_0 showing stable (s) and unstable (u) equilibria (Image by MIT OpenCourseWare.)

$$J \frac{d^2}{dt^2}(\omega_m t + \gamma_0 + \gamma') = T_0 + T' - MI^2 \sin(2\gamma_0 + 2\gamma')$$

linearization gives

$$J \frac{d^2 \gamma'}{dt^2} + (2MI^2 \cos 2\gamma_0) \gamma' = T'$$

where the constant terms cancel out by virtue of (13). With $T' = \tau_0 u_0(t)$ and initial rest conditions, the initial conditions are

$$\frac{d\gamma'}{dt}(0^+) = \frac{\tau_0}{J}$$

$$\gamma'(0^+) = 0$$

D

The solution for $\gamma'(t)$ is

$$\gamma'(t) = \frac{\tau_0}{J\sqrt{\frac{2MI^2 \cos 2\gamma_0}{J}}} \sin \left[\sqrt{\frac{2MI^2 \cos 2\gamma_0}{J}} t \right]$$

E

For stability $\cos 2\gamma_0 > 0$ as shown in Fig. 2.