6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2009
Problem Set 4 - Solutions	
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Problem 4.1

A

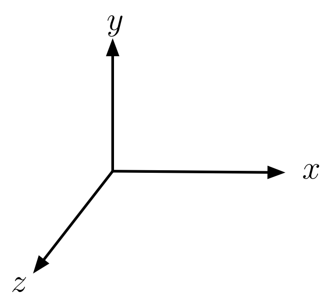

Figure 1: Cartesian coordinate axes (Image by MIT OpenCourseWare.)

For region I, we use q and q'

$$\Phi = \left(\frac{q}{4\pi\varepsilon_1 (x^2 + (y-d)^2 + z^2)^{1/2}} + \frac{q'}{4\pi\varepsilon_1 (x^2 + (y+d)^2 + z^2)^{1/2}} \right)$$

$$\begin{aligned} E_I &= -\nabla\Phi \\ &= \frac{1}{4\pi\varepsilon_1}\left(\frac{q(x\hat{i}_x + (y-d)\hat{i}_y + z\hat{i}_z)}{(x^2 + (y-d)^2 + z^2)^{3/2}} + \frac{q'(x\hat{i}_x + (y+d)\hat{i}_y + z\hat{i}_z)}{(x^2 + (y+d)^2 + z^2)^{3/2}} \right) \end{aligned}$$

For region II, use q''

$$\Phi = \frac{q''}{4\pi\varepsilon_2 (x^2 + (y-d)^2 + z^2)^{1/2}}$$

$$\begin{aligned} E_{II} &= -\nabla\Phi \\ &= \frac{q''(x\hat{i}_x + (y-d)\hat{i}_y + z\hat{i}_z)}{4\pi\varepsilon_2 (x^2 + (y-d)^2 + z^2)^{3/2}} \end{aligned}$$

B

Tangential E components are equal:

1

$$\left. \begin{aligned} \hat{n} \times [\overline{E_I} - \overline{E_{II}}] = 0 \\ E_{xI} = E_{xII} \\ E_{zI} = E_{zII} \end{aligned} \right] \text{ at } y = 0$$

2 Since there is no surface charge, *i.e.*, $\sigma_s = 0$.

$$\left.\begin{array}{r}\hat{n}\cdot(\varepsilon_I\overline{E_I} - \varepsilon_{II}\overline{E_{II}}) = 0\\ \varepsilon_I E_{Iy} = \varepsilon_{II}E_{IIy}\end{array}\right] \text{ at } y = 0$$

From 1, $\frac{1}{4\pi\varepsilon_I}\left(\frac{qx + q'x}{(x^2+d^2+z^2)^{3/2}}\right) = \frac{q''x}{4\pi\varepsilon_{II}(x^2+d^2+z^2)^{3/2}}$. Therefore, $\frac{q+q'}{\varepsilon_I} = \frac{q''}{\varepsilon_{II}}$.
From 2,

$$\varepsilon_1\frac{1}{4\pi\varepsilon_I}\left(\frac{-qd + q'd}{(x^2+d^2+z^2)^{3/2}}\right) = \varepsilon_{II}\frac{q''(-d)}{4\pi\varepsilon_{II}(x^2+d^2+z^2)^{3/2}}$$

$$-q + q' = -q''$$

Therefore,

$$\frac{q+q'}{\varepsilon_I} = \frac{q''}{\varepsilon_{II}} \Rightarrow q = \frac{\varepsilon_I}{\varepsilon_{II}}q'' - q'$$

$$q'' = q - q'$$

Therefore,

$$q = \frac{\varepsilon_I}{\varepsilon_{II}}(q - q') - q'$$

$$q\left(\frac{\varepsilon_{II} - \varepsilon_I}{\varepsilon_{II}}\right) = -q'\left(\frac{\varepsilon_I + \varepsilon_{II}}{\varepsilon_{II}}\right)$$

$$q = -q'\left(\frac{\varepsilon_I + \varepsilon_{II}}{\varepsilon_{II} - \varepsilon_I}\right) \Rightarrow q' = -q\frac{(\varepsilon_{II} - \varepsilon_I)}{(\varepsilon_{II} + \varepsilon_I)}$$

and

$$\begin{aligned} q &= \frac{\varepsilon_I}{\varepsilon_{II}}q'' - q' \\ &= \frac{\varepsilon_I}{\varepsilon_{II}}q'' - q + q'' \\ 2q &= q''\left(\frac{\varepsilon_I + \varepsilon_{II}}{\varepsilon_{II}}\right) \\ q &= \frac{\varepsilon_I + \varepsilon_{II}}{2\varepsilon_{II}}q'' \Rightarrow q'' = \frac{2\varepsilon_{II}q}{(\varepsilon_I + \varepsilon_{II})} \end{aligned}$$

C

$$\begin{aligned} \bar{f} &= q\bar{E}_I(x = 0, y = d, z = 0) \\ &= q\left(\frac{q'(0\hat{i}_x + 2d\hat{i}_y + 0\hat{i}_z)}{4\pi\varepsilon_I(0^2 + (2d)^2 + 0^2)^{\frac{3}{2}}}\right) \\ &= \frac{2dq'q\hat{i}_y}{4\pi\varepsilon_I\cdot 8d^3} = \frac{q'q\hat{i}_y}{4\pi\varepsilon_I\cdot 4d^2} \\ &= \frac{q\left(-q\left(\frac{\varepsilon_{II}-\varepsilon_I}{\varepsilon_I+\varepsilon_{II}}\right)\right)}{4\pi\varepsilon_I\cdot 4d^2}\hat{i}_y = \frac{-q^2(\varepsilon_{II} - \varepsilon_I)}{16\pi\varepsilon_I d^2(\varepsilon_I + \varepsilon_{II})}\hat{i}_y \\ &= \frac{q^2(\varepsilon_I - \varepsilon_{II})}{16\pi\varepsilon_I(\varepsilon_I + \varepsilon_{II})d^2}\hat{i}_y \end{aligned}$$

Problem 4.2

This is a charge relaxation problem, so we use, as shown in class, the equations (done in lecture 12)

$$\nabla \cdot \overrightarrow{J}_f + \frac{\partial \rho_f}{\partial t} = 0$$

We substitute in $\nabla \cdot \overrightarrow{E} = \frac{\rho_f}{\varepsilon}$ and $\overrightarrow{J}_f = \sigma \overrightarrow{E}$ to get

$$\sigma \nabla \cdot \overrightarrow{E} + \frac{\partial \rho_f}{\partial t} = 0 \Rightarrow \frac{\partial \rho_f}{\partial t} + \frac{\sigma}{\varepsilon}\rho_f = 0$$

So

$$\rho_f = \rho(\overrightarrow{r}, t=0)e^{-\frac{t}{\tau_e}}; \tau_e = \frac{\varepsilon}{\sigma}$$

Thus

$$\rho_f(r,t) = \begin{cases} \frac{\rho_0 r}{a_0} e^{-\frac{t}{\tau_e}} & 0 < r < a_0 \\ 0 & r > a_0 \end{cases}$$

Notice: $\rho_f(r,t)$ is 0 for $r > a_0$. Nonetheless, there is still conduction and displacement current for $a_0 < r < a_1$. By Gauss, $\oint_S \varepsilon \overrightarrow{E} \cdot d\overrightarrow{a} = \int_V \rho dV$. Choosing S as a cylinder with radius r

$$\oint_S \varepsilon \overrightarrow{E} \cdot d\overrightarrow{a} = \varepsilon E_r r 2\pi L$$

where L is the length of the cylinder. Note that $\overrightarrow{E} \cdot d\overrightarrow{a} = 0$ on cylinder ends.

Now for RHS of Gauss

$$\int_V \rho_f dV = \int_0^L \int_0^r \int_0^{2\pi} \rho_f(r',t)r' d\phi dr' dz = 2\pi L \int_0^r \rho_f(r',t)r' dr'$$

$r < a_0$:

$$\int_V \rho_f dV = 2\pi L \int_0^r \frac{\rho_0 (r')^2}{a_0} e^{-\frac{t}{\tau_e}} dr'$$

$$= 2\pi L \frac{\rho_0}{a_0} \frac{r^3}{3} e^{-\frac{t}{\tau_e}}$$

$a_0 < r < a_1$:

$$\int_v \rho_f dV = 2\pi L \int_0^{a_0} \frac{\rho_0 (r')^2}{a_0} e^{-\frac{t}{\tau_e}} dr'$$

$$= 2\pi L \frac{\rho_0}{a_0} e^{-\frac{t}{\tau_e}} \frac{a_0^3}{3} = \frac{2\pi L}{3} \rho_0 a_0^2 e^{-t/\tau_e}$$

$r > a_1$:

$$\int_V \rho_f dV = 2\pi L \int_0^{a_0} \frac{\rho_0 (r')^2}{a_0} dr'$$

$$= 2\pi L \frac{\rho_0}{a_0} \frac{a_0^3}{3} = \frac{2\pi L}{3} \rho_0 a_0^2$$

(total charge on sphere is constant equal to total initial charge including surface charge at $r = a_1$)

So:

$$\overrightarrow{E} = \begin{cases} \frac{\rho_0 r^2}{3a_0\varepsilon} e^{-\frac{t}{\tau_e}} \hat{i}_r & r < a_0 \\ \frac{\rho_0 a_0^2}{3r\varepsilon} e^{-\frac{t}{\tau_e}} \hat{i}_r & a_0 < r < a_1 \\ \frac{\rho_0 a_0^2}{3r\varepsilon_0} \hat{i}_r & r > a_1 \end{cases}$$

$$\sigma_{sf} = \varepsilon_0 E_r(r = a_1^+) - \varepsilon E_r(r = a_1^-)$$

$$\sigma_{sf} = \frac{\rho_0 a_0^2}{3a_1}(1 - e^{-\frac{t}{\tau_e}})$$

Problem 4.3

A

There are no surface currents, so we have continuity of normal $\overrightarrow{B}$ and tangential $\overrightarrow{H}$. Also, if $\mu \to \infty$, $\overrightarrow{H} = 0$ inside, but $\overrightarrow{B}$ may still be nonzero. Equivalent image problem:

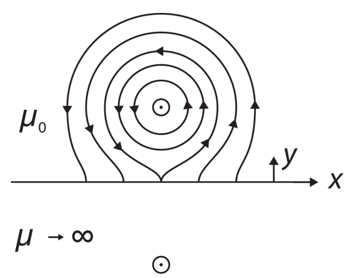

Figure 2: Magnetic field lines due to a line current above an infinitely magnetically permeable region (Image by MIT OpenCourseWare.)

Boundary conditions: $H_x = H_z = 0$ at $y = 0$

B

Assume line current at origin:

$$\int \overrightarrow{H} \cdot d\overrightarrow{l} = I \Rightarrow H_\phi = \frac{I}{2\pi r}$$

$$\nabla \times \overrightarrow{A} = \overrightarrow{B} \Rightarrow -\frac{\partial A_z}{\partial r} = \frac{I\mu_0}{2\pi r}$$

Suggesting: $A_z = -\frac{I\mu_0}{2\pi}\ln(r) + \text{constant}$. Assume line current at $y = d$:

$$A_z = -\frac{I\mu_0}{2\pi} \ln\sqrt{(y-d)^2 + x^2}$$

Now 2 line currents; one at $y = d$ and one at $y = -d$.

$$A_z = -\frac{I\mu_0}{2\pi}\left\{\ln\left[\sqrt{x^2 + (y-d)^2}\right] + \ln\left[\sqrt{x^2 + (y+d)^2}\right]\right\}$$

C

$$\frac{1}{\mu_0}\nabla \times \overrightarrow{A} = \overrightarrow{H}$$

$$= \frac{1}{\mu_0}\left(\hat{i}_x \frac{\partial A_z}{\partial y} - \hat{i}_y \frac{\partial A_z}{\partial x}\right)$$

$$\overrightarrow{H} = -\frac{I}{2\pi}\left[\frac{(y-d)\hat{i}_x - x\hat{i}_y}{x^2+(y-d)^2} + \frac{(y+d)\hat{i}_x - x\hat{i}_y}{x^2+(y+d)^2}\right]$$

D

Field line equation:

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -\frac{\frac{x}{x^2+(y-d)^2} + \frac{x}{x^2+(y+d)^2}}{\frac{y-d}{x^2+(y-d)^2} + \frac{y+d}{x^2+(y+d)^2}} = \frac{-\frac{\partial A_z}{\partial x}}{\frac{\partial A_z}{\partial y}}$$

$$\frac{\partial A_z}{\partial y}dy = -\frac{\partial A_z}{\partial x}dx \Rightarrow \frac{\partial A_z}{\partial x}dx + \frac{\partial A_z}{\partial y}dy = dA_z = 0$$

$$A_z = \text{constant} \Rightarrow \left[x^2+(y-d)^2\right]\left[x^2+(y+d)^2\right] = \text{constant}$$

E

$$\frac{\overrightarrow{F}}{\text{unit length}} = \underbrace{\overrightarrow{I}}_{\substack{\text{current at}\\ \text{y=d}}} \times \underbrace{\overrightarrow{B}}_{\substack{\overrightarrow{B}\text{ field}\\ \text{caused by image current}\\ \text{alone at } x=0,\, y=-d}}$$

$$\frac{\overrightarrow{F}}{\text{unit length}} = (I\hat{i}_z) \times \left(\frac{-\mu_0 I}{2\pi}\right)\left(\frac{(y+d)\hat{i}_x - x\hat{i}_y}{x^2+(y+d)^2}\right)\Bigg|_{x=0,y=d}$$

$$= (I\hat{i}_z) \times \left(\frac{-\mu_0 I}{2\pi}\right)\left(\frac{2d}{(2d)^2}\hat{i}_x\right)$$

$$\frac{\overrightarrow{F}}{\text{unit length}} = -\frac{\mu_0 I^2}{4\pi d}\hat{i}_y$$

Problem 4.4

A

$\nabla \cdot \overrightarrow{J} = 0;$ by symmetry we just have x component of $\overline{J}$

$$\frac{\partial J_x}{\partial x} = 0 \Rightarrow \overrightarrow{J} = J_0\hat{i}_x, J_0 \text{ is constant}$$

$$E_x \cdot \sigma_x = J_x; \quad \sigma_x = \sigma_0 e^{-\frac{x}{s}}$$

$$E_x = \frac{J_x}{\sigma_x} = \frac{J_0}{\sigma_0 e^{-\frac{x}{s}}} = \frac{J_0}{\sigma_0}e^{\frac{x}{s}}$$

$$V_0 = \int_0^s E_x dx = \frac{J_0}{\sigma_0}\int_0^s e^{\frac{x}{s}}dx = \frac{J_0 s}{\sigma_0}e^{\frac{x}{s}}\Big|_0^s = \frac{J_0 s}{\sigma_0}(e-1)$$

$$I_0 = J_0 \cdot ld$$

$$R = \frac{V_0}{I_0} = \frac{\frac{J_0 s}{\sigma_0}(e-1)}{J_0 ld} = \frac{s}{\sigma_0 ld}(e-1)$$

B

$$\nabla \cdot (\varepsilon \overline{E}) = \rho \Rightarrow \rho = \varepsilon \frac{\partial E_x}{\partial x}$$

$$\rho = \frac{J_0 \varepsilon}{\sigma_0 s} e^{\frac{x}{s}}$$

At $x = 0$:

$$\sigma_s = \varepsilon E_x \big|_{x=0} = \frac{\varepsilon J_0}{\sigma_0}$$

At $x = s$:

$$\sigma_s = -\varepsilon E_x \big|_{x=s} = -\frac{\varepsilon J_0}{\sigma_0} e$$

C

$$q_V = ld \int_0^s \rho dx = \frac{ldJ_0\varepsilon}{\sigma_0}(e-1)$$

Total surface charge

$$q_S = (\sigma_s|_{x=s} + \sigma_s|_{x=0}) ld = -\frac{ldJ_0\varepsilon}{\sigma_0}(e-1) = -q_V$$

$$q_S + q_V = 0$$

Problem 4.5

A

As no volume charge in the dielectric

$$\nabla \cdot \bar{D} = 0$$

From symmetry we just have r component

$$\frac{1}{r^2}\frac{\partial (r^2 D_r)}{\partial r} = 0 \Rightarrow D_r = \frac{A}{r^2}, \varepsilon(r) = \frac{\varepsilon_1 r}{a}$$

$$D_r = \varepsilon E_r \Rightarrow E_r = \frac{D_r}{\varepsilon(r)} = \frac{Aa}{r^2 \varepsilon_1 r} = \frac{Aa}{\varepsilon_1 r^3}$$

$$v = \int_a^b E_r dr = \int_a^b \frac{Aa}{\varepsilon_1 r^3} dr = \frac{-Aa}{2\varepsilon_1}\frac{1}{r^2}\bigg|_a^b = \frac{Aa}{2\varepsilon_1}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

$$A = \frac{2\varepsilon_1 v}{a}\frac{1}{\frac{1}{a^2}-\frac{1}{b^2}}, \quad E_r = \frac{2v}{\frac{1}{a^2}-\frac{1}{b^2}}\frac{1}{r^3}$$

$$\bar{E} = -\nabla \Phi \Rightarrow E_r = -\frac{\partial \Phi}{\partial r} = \frac{2v}{\frac{1}{a^2}-\frac{1}{b^2}}\frac{1}{r^3}$$

$$\Phi = \int -E_r dr = +\frac{v}{\frac{1}{a^2}-\frac{1}{b^2}}\frac{1}{r^2}$$

B

$$\sigma_s|_{r=a} = \varepsilon(r)E_r|_{r=a_+} = \frac{2\varepsilon_1 v}{\frac{1}{a^2} - \frac{1}{b^2}} \frac{1}{a^3}$$

$$\sigma_s|_{r=b} = -\varepsilon(r)E_r|_{r=b_-} = \frac{-2\varepsilon_1 v \frac{b}{a}}{\frac{1}{a^2} - \frac{1}{b^2}} \frac{1}{b^3} = \frac{-2\varepsilon_1 v}{\frac{1}{a^2} - \frac{1}{b^2}} \frac{1}{ab^2}$$

C

$$q = 4\pi a^2 \sigma_s|_{r=a_+} = -4\pi b^2 \sigma_s|_{r=b_-} = \frac{8\pi\varepsilon_1 v}{\left(\frac{1}{a^2} - \frac{1}{b^2}\right) a}$$

$$C = \frac{q}{v} = \frac{8\pi\varepsilon_1}{\left(\frac{1}{a^2} - \frac{1}{b^2}\right) a}$$