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6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2009
Problem Set 2 - Solutions	
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Problem 2.1

A

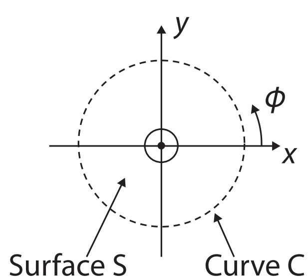

Figure 1: Surface S and contour C for using Ampere's law (Image by MIT OpenCourseWare.)

Step 1: Find field of z-directed line current, $\overrightarrow{I}$, at $x = y = 0$.

$$\overrightarrow{I} = I\hat{i}_z$$

By symmetry: $\overrightarrow{H} = H_\phi \hat{i}_\phi$ in cylindrical coordinates. By Ampere:

$$\oint_C \overrightarrow{H} \cdot d\overrightarrow{l} = \underbrace{\int_S \overrightarrow{J} \cdot d\overrightarrow{a}}_{\text{current going through } S}$$

$$(2\pi r)H_\phi = I$$

$$H_\phi = \frac{I}{2\pi r}\hat{i}_\phi \tag{1}$$

$$\hat{i}_\phi = \frac{-y}{\sqrt{x^2+y^2}}\hat{i}_x + \frac{x}{\sqrt{x^2+y^2}}\hat{i}_y$$

$$\overrightarrow{H} = \frac{I}{2\pi(x^2+y^2)}(-y\hat{i}_x + x\hat{i}_y)$$

Step 2: Find solution by adding two translated $\overrightarrow{H}$ fields:

$$\overrightarrow{H}_{\text{total}} = \overrightarrow{H}_{I_1} + \overrightarrow{H}_{I_2}$$

$$= \frac{I_1}{2\pi\left[x^2 + (y - \frac{d}{2})^2\right]}\left[-(y - \frac{d}{2})\hat{i}_x + x\hat{i}_y\right] + \frac{I_2}{2\pi\left[x^2 + (y + \frac{d}{2})^2\right]}\left[-(y + \frac{d}{2})\hat{i}_x + x\hat{i}_y\right]$$

B

Step 3: We want field in $y = 0$ plane so $y \to 0$.

i

$$I_1 = I, I_2 = 0$$

$$\overrightarrow{H}_{\text{tot}} = \frac{I}{2\pi(x^2 + \frac{d^2}{4})}\left[\frac{d}{2}\hat{i}_x + x\hat{i}_y\right]$$

ii

$$I_1 = I, I_2 = I$$

$$\overrightarrow{H}_{\text{tot}} = \frac{I}{\pi(x^2 + \frac{d^2}{4})}\left[x\hat{i}_y\right]$$

iii

$$I_1 = I, I_2 = -I$$

$$\overrightarrow{H}_{\text{tot}} = \frac{I}{2\pi(x^2 + \frac{d^2}{4})}[d\hat{i}_x]$$

C

$$\overrightarrow{F} = q\overrightarrow{v} \times (\mu_0\overrightarrow{H})$$

so

$$d\overrightarrow{F} = dq\left[\overrightarrow{v} \times \left(\mu_0\overrightarrow{H}\right)\right]$$

In our problem the line current is a moving line charge, so $dq = \lambda dl$

$$d\overrightarrow{F} = \lambda\overrightarrow{v} \times \left[\mu_0\overrightarrow{H}\right]dl = \bar{I} \times (\mu_0\overrightarrow{H})dl$$

$$\overrightarrow{F} = \int_0^L \overrightarrow{I} \times (\mu_0\overrightarrow{H})dl = (\overrightarrow{I} \times \mu_0\overrightarrow{H})L$$

So

$$\frac{\text{Force}}{\text{Length}} = \overrightarrow{I} \times (\mu_0\overrightarrow{H})$$

We don't need to know $\overrightarrow{H_{I_1}}$ since I_1 cannot exert a net force on itself. So we need a field of I_2

$$\overrightarrow{H_{I_2}} = \frac{I_2\left[-\left(y + \frac{d}{2}\right)\hat{i}_x + x\hat{i}_y\right]}{2\pi\left(x^2 + \left(y + \frac{d}{2}\right)^2\right)}$$

I_1 is at $x = 0, y = \frac{d}{2}$, so $x \to 0, y \to \frac{d}{2}$ above.

$$\overrightarrow{H} = \frac{-I_2 d}{2\pi d^2}\hat{i}_x = \frac{-I_2}{2\pi d}\hat{i}_x$$

$$\frac{\text{Force}}{\text{Length}} = (\overrightarrow{I_1}) \times (\mu_0 \overrightarrow{H}) = (I_1 \hat{i}_z) \times \left(\frac{-\mu_0 I_2}{2\pi d}\right) \hat{i}_x$$

$$= \frac{-\mu_0 I_1 I_2}{2\pi d} \hat{i}_y$$

$(i) \qquad I_1 = I, \quad I_2 = 0$

$$\frac{\text{Force}}{\text{Length}} = 0$$

$(ii) \qquad I_1 = I, \quad I_2 = I$

$$\frac{\text{Force}}{\text{Length}} = \frac{-\mu_0 I^2}{2\pi d} \hat{i}_y$$

$(iii) \qquad I_1 = I, \quad I_2 = -I$

$$\frac{\text{Force}}{\text{Length}} = +\frac{\mu_0 I^2}{2\pi d} \hat{i}_y$$

Problem 2.2

A

The idea here is similar to applying the chain rule in a 1D problem

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \left[\frac{d}{df}\left(\frac{1}{f(x)}\right)\right]\left[\frac{df}{dx}\right] = \frac{-f'(x)}{f^2(x)}$$

$f(x)$ corresponds to $|\overrightarrow{r} - \overrightarrow{r}'|$. So, by diff. $f(x)$ we get part of the answer to the derivative of $\frac{1}{f(x)}$. But we can just do it directly too.

$$|\overrightarrow{r} - \overrightarrow{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\nabla\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}'|}\right] = \hat{i}_x \frac{\partial}{\partial x}\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}'|}\right] + \hat{i}_y \frac{\partial}{\partial y}\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}'|}\right] + \hat{i}_z \frac{\partial}{\partial z}\left[\frac{1}{|\overrightarrow{r} - \overrightarrow{r}'|}\right]$$

So we can apply the trick above by just considering x, y, and z components separately.

$$\frac{\partial}{\partial x}|\overrightarrow{r} - \overrightarrow{r}'| = \frac{\partial}{\partial x}\left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)$$

$$= \frac{x-x'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

$$= \frac{x-x'}{|\overrightarrow{r} - \overrightarrow{r}'|}$$

Similarly, $\frac{\partial}{\partial y}|\overrightarrow{r} - \overrightarrow{r}'| = \frac{y-y'}{|\overrightarrow{r} - \overrightarrow{r}'|}$ and $\frac{\partial}{\partial z}|\overrightarrow{r} - \overrightarrow{r}'| = \frac{z-z'}{|\overrightarrow{r} - \overrightarrow{r}'|}$.

$$\frac{\partial}{\partial x}\left(\frac{1}{|\overrightarrow{r} - \overrightarrow{r}'}\right) = \frac{-\frac{\partial}{\partial x}|\overrightarrow{r} - \overrightarrow{r}'|}{|\overrightarrow{r} - \overrightarrow{r}'|^2}$$

and so on for y and z.

$$|\overrightarrow{r} - \overrightarrow{r}'|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

so:

$$\nabla\left[\frac{1}{|\overrightarrow{r}-\overrightarrow{r}'|}\right]=\frac{-\left[(x-x')\hat{i}_x+(y-y')\hat{i_y}+(z-z')\hat{i_z}\right]}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}$$

Denominators $= |\overrightarrow{r}-\overrightarrow{r}'|^{\frac{3}{2}}$. Thus,

$$\nabla\left[\frac{1}{|\overrightarrow{r}-\overrightarrow{r}'|}\right]=\frac{-(\overrightarrow{r}-\overrightarrow{r}')}{|\overrightarrow{r}-\overrightarrow{r}'|^3}=\frac{-1}{|\overrightarrow{r}-\overrightarrow{r}'|^2}\frac{(\overrightarrow{r}-\overrightarrow{r}')}{|\overrightarrow{r}-\overrightarrow{r}'|}$$

$$=\frac{-\hat{i}_{r'r}}{|\overrightarrow{r}-\overrightarrow{r}'|^2}$$

B

Follows from (A) immediately by substitution. Remember ∇ is derived in terms of unprimed x, y, z. ∇ does not affect x', y', z'.

C

$$\Phi(\overrightarrow{r})=\int_{V'}\frac{\rho(\overrightarrow{r}')dV'}{4\pi\varepsilon_0|\overrightarrow{r}-\overrightarrow{r}'|}$$

$\rho(\overrightarrow{r}')$ = charge density in $\frac{C}{m^3}$. We have λ in units of $\frac{C}{m}$. In this sense, $\rho\rightarrow\infty$ at the ring. We can represent this in cylindrical coordinates by $\rho(\overrightarrow{r}')=\lambda_0\delta(z)\delta(r-a)$. Then we can evaluate the triple integral

$$\int\int\int\frac{\lambda_0\delta(z)\delta(r-a)rdrd\phi dz}{4\pi\epsilon_0|\overrightarrow{r}-\overrightarrow{r}'|}=\int_0^{2\pi}\frac{\lambda_0 ad\phi}{4\pi\epsilon_0|\overline{r}-\overline{r}'|}$$

But, we can skip that unnecessary work by simply considering infinitesimal charges $(ad\phi)\lambda_0$ around the ring.

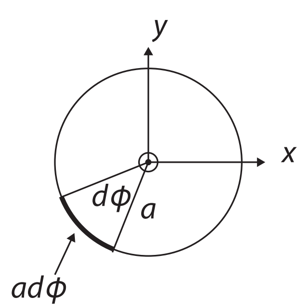

Figure 2: A ring of line charge with infinitesimal charge elements $dq=\lambda_0 ad\phi$. (Image by MIT OpenCourseWare.)

We only care about z axis in this problem as well, so, by symmetry, there is no field in the x and y directions.

$$\Phi(\vec{r}) = \int_0^{2\pi} \frac{\lambda_0(ad\phi)}{4\pi\varepsilon_0 \underbrace{(a^2+z^2)^{\frac{1}{2}}}_{\substack{\text{distance from} \\ \text{the charge} \\ \text{element } \lambda_0 ad\phi \\ \text{to the point } z \\ \text{on the } z\text{-axis}}}}$$

$$\Phi(\vec{r}) = \frac{\lambda_0 a}{2\varepsilon_0 (a^2+z^2)^{\frac{1}{2}}}$$

on the z-axis.

$$\vec{E} = -\nabla\Phi(\vec{r}) = -\left(\hat{i}_x \overbrace{\frac{\partial}{\partial x}\Phi}^{0} + \hat{i}_y \overbrace{\frac{\partial}{\partial y}\Phi}^{0} + \hat{i}_z \frac{\partial}{\partial z}\Phi\right)$$

$$\vec{E} = -\hat{i}_z \frac{\partial}{\partial z}\left(\frac{\lambda_0 a}{2\varepsilon_0 (a^2+z^2)^{\frac{1}{2}}}\right)$$

$$\vec{E} = \hat{i}_z \frac{a\lambda_0 z}{2\varepsilon_0 (a^2+z^2)^{\frac{3}{2}}}$$

Using the equation from the Problem 2.2 Statement with z component only (symmetry) and with $\rho(\vec{r}\,')dV' \to \lambda_0 ad\phi$

$$E_z(z) = \int_0^{2\pi} \frac{\lambda_0 ad\phi \cos\theta}{4\pi\varepsilon_0 (z^2+a^2)}, \cos\theta = \frac{z}{(a^2+z^2)^{\frac{1}{2}}}$$

$$= \int_0^{2\pi} \frac{\lambda_0 az}{(a^2+z^2)^{\frac{3}{2}}} \frac{d\phi}{4\pi\varepsilon_0}$$

$$= \frac{\lambda_0 az}{2\varepsilon_0 (a^2+z^2)^{\frac{3}{2}}}$$

Limit $|z| \to \infty$

$$\sqrt{a^2+z^2} \to |z|$$

$$\Phi(z) \approx \frac{\lambda_0 a}{2\varepsilon_0 (a^2+z^2)^{\frac{1}{2}}} \approx \frac{2\pi\lambda_0 a}{4\pi\varepsilon_0 |z|} \approx \frac{Q}{4\pi\varepsilon_0 |z|}$$

$Q = 2\pi\lambda_0 a$ (total charge on loop). $\Phi(z)$ looks like potential from point charge in far field.

$$E_z = \frac{\lambda_0 az}{2\varepsilon_0 (a^2+z^2)^{\frac{3}{2}}} \approx \frac{\lambda_0 az}{2\varepsilon_0 |z|^3} = \begin{cases} \frac{Q}{4\pi\varepsilon_0 |z|^2} & z > 0 \\ \frac{-Q}{4\pi\varepsilon_0 |z|^2} & z < 0 \end{cases}$$

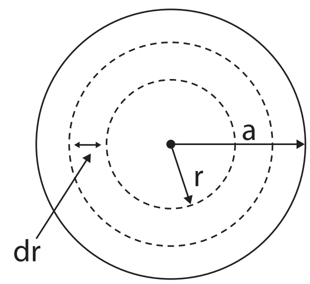

Figure 3: A line charge ring of width dr in the disk (Image by MIT OpenCourseWare.)

D

From (C), $\Phi = \frac{\lambda_0 r}{2\varepsilon_0 (r^2+z^2)^{\frac{1}{2}}}$ for a ring of radius r. But now we have σ_0, not λ_0. How do we express λ_0 in terms of σ_0? Take a ring of width dr in the disk (see figure). Total charge in the ring $= \underbrace{(r)(2\pi)}_{\text{circumference}}(dr)\sigma_0$.

Line charge density $= \lambda_0 = \frac{\text{total charge}}{\text{length}} = \sigma_0 dr$
So: $\lambda_0 = \sigma_0 dr$

$$d\Phi = \frac{\sigma_0 r dr}{2\varepsilon_0 (r^2+z^2)^{\frac{1}{2}}}$$

$$\Phi_{\text{total}} = \int_0^a \frac{\sigma_0 r dr}{2\varepsilon_0 (r^2+z^2)^{\frac{1}{2}}} = \frac{\sigma_0}{2\varepsilon_0}\int_0^a \frac{r dr}{(r^2+z^2)^{\frac{1}{2}}}$$

$$= \frac{\sigma_0}{2\varepsilon_0}\left[\sqrt{r^2+z^2}\right]\Big|_{r=0}^{r=a} = \frac{\sigma_0}{2\varepsilon_0}\left[\sqrt{a^2+z^2} - |z|\right]$$

$$\overrightarrow{E} = -\nabla\Phi_{\text{total}} = \frac{\sigma_0}{2\varepsilon_0} z \left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{a^2+z^2}}\right]\overline{i_z}$$

E

As $z \to \infty$,

$$(a^2+z^2)^{\frac{1}{2}} \to |z| + \frac{a^2}{2|z|}; \quad (a^2+z^2)^{-\frac{1}{2}} \to \frac{1}{|z|}\left(1 - \frac{a^2}{2z^2}\right)$$

$$\Phi_{\text{total}} \to \frac{\pi a^2 \sigma_0}{4\epsilon_0 \pi |z|}$$

$$\bar{E} \to \frac{\pi a^2 \sigma_0}{4\pi\varepsilon_0 z^2}\overline{i_z}$$

just like a point charge of $\sigma_0 \pi a^2$.

F

As $a \to \infty$, z in the $\sqrt{a^2+z^2}$ can be neglected, so

$$\Phi_{\text{total}} \to \frac{\sigma_0}{2\varepsilon_0}\left[a - |z|\right]$$

$$E_z \to \frac{\sigma_0 z}{2\varepsilon_0}\left[\frac{1}{|z|} - 0\right] = \begin{cases} \frac{\sigma_0}{2\varepsilon_0} & z > 0 \\ \frac{-\sigma_0}{2\varepsilon_0} & z < 0 \end{cases}$$

just like a sheet charge.

Problem 2.3

A

By the divergence theorem:

i

$$\int_V \nabla \cdot (\nabla \times \overrightarrow{A}) dV = \oint_S (\nabla \times \overrightarrow{A}) \cdot d\overrightarrow{a}$$

where S encloses V.

ii By Stokes' Theorem:

$$\int_{S'} (\nabla \times \overrightarrow{A}) \cdot d\overrightarrow{a} = \oint_C \overrightarrow{A} \cdot d\overrightarrow{l}$$

Suppose S is as in figure 4

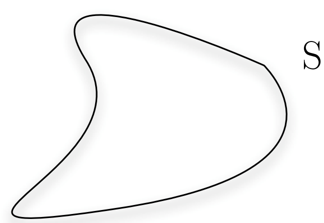

Figure 4: Closed surface S (Image by MIT OpenCourseWare.)

and S' is as in figure 5

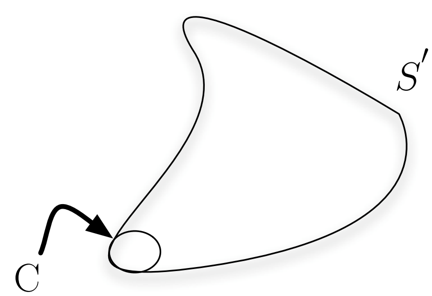

Figure 5: Open surface S' bounded by contour C (Image by MIT OpenCourseWare.)

i.e., S' is the same as S, except for the contour curve C, which makes S' slightly unclosed. Now consider limit as $C \to 0$ (Figure 6)

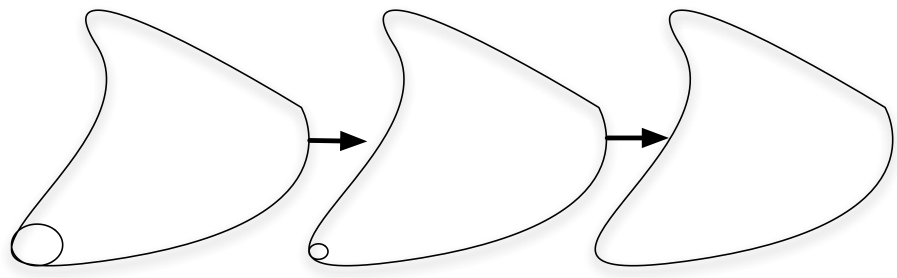

Figure 6: Limit as $C \to 0$ (Image by MIT OpenCourseWare.)

In limit $C \to 0, S' \to S$. If C is 0, then $\oint_C \overrightarrow{A} \cdot d\overrightarrow{l} = 0$. By equation (ii), $\oint_S (\nabla \times \overrightarrow{A}) \cdot d\overrightarrow{a} = 0$. By equation (i), $\int_V \nabla \cdot (\nabla \times \overrightarrow{A}) dV = 0$. Since V can be any volume, argument of integral must be identically 0.

$\nabla \cdot (\nabla \times \overrightarrow{A}) = 0$

B

$$\overrightarrow{A} = A_x \hat{i}_x + A_y \hat{i}_y + A_z \hat{i}_z$$

$$\nabla \cdot (\nabla \times \overrightarrow{A}) = \nabla \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{i}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{i}_z \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$= 0$ because of interchangeability of partial derivatives

In cylindrical coordinates

$$\bar{A} = A_r \bar{i}_r + A_\phi \bar{i}_\phi + A_z \bar{i}_z$$

$$\nabla \times \bar{A} = \bar{i}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \bar{i}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \bar{i}_z \frac{1}{r} \left[\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla \cdot (\nabla \times \bar{A}) = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\partial A_z}{\partial \phi} - r \frac{\partial A_\phi}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \frac{\partial}{\partial z} \left[\frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \phi} \right]$$

$$= \frac{1}{r} \frac{\partial^2 A_z}{\partial r \partial \phi} - \frac{r}{r} \frac{\partial^2 A_\phi}{\partial r \partial z} - \frac{1}{r} \frac{\partial A_\phi}{\partial z} + \frac{1}{r} \frac{\partial^2 A_r}{\partial \phi \partial z} - \frac{1}{r} \frac{\partial^2 A_z}{\partial r \partial \phi} + \frac{\partial^2 A_\phi}{\partial r \partial z} + \frac{1}{r} \frac{\partial A_\phi}{\partial z} - \frac{1}{r} \frac{\partial^2 A_r}{\partial \phi \partial z}$$

$= 0$

Problem 2.4

(Zahn, Problem 23, Chapter 1)

A

Cartesian	Cylindrical	Spherical
$h_x = 1$	$h_r = 1$	$h_r = 1$
$h_y = 1$	$h_\phi = r$	$h_\theta = r$
$h_z = 1$	$h_z = 1$	$h_\phi = r \sin\theta$

B

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw$$
$$= \nabla f \cdot \overline{d\ell}$$
$$= \nabla f \cdot [h_u du \bar{i}_u + h_v dv \bar{i}_v + h_w dw \bar{i}_w]$$
$$(\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u}; \quad (\nabla f)_v = \frac{1}{h_v}\frac{\partial f}{\partial v}; \quad (\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w}$$
$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\bar{i}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\bar{i}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\bar{i}_w$$

c)

$$dS_u = h_v h_w dv dw; \quad dS_v = h_u h_w du dw; \quad dS_w = h_u h_v du dv$$
$$dV = h_u h_v h_w du dv dw$$

D

Divergence

$$\Phi = \oint_S \bar{A} \cdot \overline{dS} = \int_{1,u} A_u h_v h_w dv dw - \int_{1',u-\Delta u} A_u h_v h_w dv dw$$
$$+ \int_{2,v+\Delta v} A_v h_u h_w du dw - \int_{2',v} A_v h_u h_w du dw$$
$$+ \int_{3,w+\Delta w} A_w h_u h_v du dv - \int_{3',w} A_w h_u h_v du dv$$
$$= \left\{ \frac{A_u h_v h_w|_u - A_u h_v h_w|_{u-\Delta u}}{\Delta u} + \frac{A_v h_u h_w|_{v+\Delta v} - A_v h_u h_w|_v}{\Delta v} + \frac{A_w h_u h_v|_{w+\Delta w} - A_w h_u h_v|_w}{\Delta w} \right\} \Delta u \Delta v \Delta w$$
$$\nabla \cdot \bar{A} = \lim_{\Delta u \to 0, \Delta v \to 0, \Delta w \to 0} \frac{\oint_S \bar{A} \cdot \overline{dS}}{\Delta V} = \frac{\oint_S \bar{A} \cdot \overline{dS}}{h_u h_v h_w \Delta u \Delta v \Delta w}$$
$$= \frac{1}{h_u h_v h_w}\left[\frac{\partial(h_v h_w A_u)}{\partial u} + \frac{\partial(h_u h_w A_v)}{\partial v} + \frac{\partial(h_u h_v A_w)}{\partial w}\right]$$

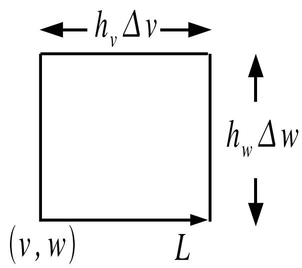

Figure 7: Surface for determining $(\nabla \times \bar{A})_u$ (Image by MIT OpenCourseWare.)

Curl

$$(\nabla \times \bar{A})_u = \lim_{\Delta v \to 0, \Delta w \to 0} \frac{\oint_L \bar{A} \cdot \overline{d\ell}}{h_v h_w \Delta v \Delta w}$$

$$\oint_L \bar{A} \cdot \overline{d\ell} = [A_v h_v \Delta v|_w - A_v h_v \Delta v|_{w+\Delta w}] + [A_w h_w \Delta w|_{v+\Delta v} - A_w h_w \Delta w|_v]$$

$$(\nabla \times \bar{A})_u = \lim_{\Delta v \to 0, \Delta w \to 0} \frac{1}{h_v h_w} \left\{ \frac{A_v h_v|_w - A_v h_v|_{w+\Delta w}}{\Delta w} + \frac{A_w h_w|_{v+\Delta v} - A_w h_w|_v}{\Delta v} \right\}$$

$$= \frac{1}{h_v h_w} \left[\frac{\partial (h_w A_w)}{\partial v} - \frac{\partial (h_v A_v)}{\partial w} \right]$$

Similarly

$$(\nabla \times \bar{A})_v = \frac{1}{h_u h_w} \left[\frac{\partial (h_u A_u)}{\partial w} - \frac{\partial (h_w A_w)}{\partial u} \right]$$

$$(\nabla \times \bar{A})_w = \frac{1}{h_u h_v} \left[\frac{\partial (h_v A_v)}{\partial u} - \frac{\partial (h_u A_u)}{\partial v} \right]$$

E

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_u h_w}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial f}{\partial w} \right) \right]$$