6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2009
Problem Set 1 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 1.1

A

$$\overrightarrow{F} = q(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}) \quad \text{Lorentz Force Law}$$

In the steady state $\overrightarrow{F} = 0$, so

$$q\overrightarrow{E} = -q\overrightarrow{v} \times \overrightarrow{B} \Rightarrow \overrightarrow{E} = -\overrightarrow{v} \times \overrightarrow{B}$$

$$\overrightarrow{v} = \begin{cases} v_y \hat{i}_y & \text{pos. charge carriers} \\ -v_y \hat{i}_y & \text{neg. charge carriers} \end{cases}$$

$$\overrightarrow{B} = B_0 \hat{i}_z$$

so

$$\overrightarrow{E} = \begin{cases} -v_y B_0 \hat{i}_x & \text{pos. charge carriers} \\ v_y B_0 \hat{i}_x & \text{neg. charge carriers} \end{cases}$$

B

$$v_H = \Phi(x = d) - \Phi(x = 0) = -\int_0^d E_x dx = \int_d^0 E_x dx$$

$$v_H = \begin{cases} v_y B_0 d & \text{pos. charges} \\ -v_y B_0 d & \text{neg. charges} \end{cases}$$

C

As seen in part (b), positive and negative charge carriers give opposite polarity voltages, so answer is "yes."

Problem 1.2

By problem

$$\rho = \begin{cases} \frac{\rho_b r}{b}; & r < b \\ \rho_a; & b < r < a \end{cases}$$

Also, no σ_s at $r = b$, but non zero σ_s such that $\vec{E} = 0$ for $r > a$.

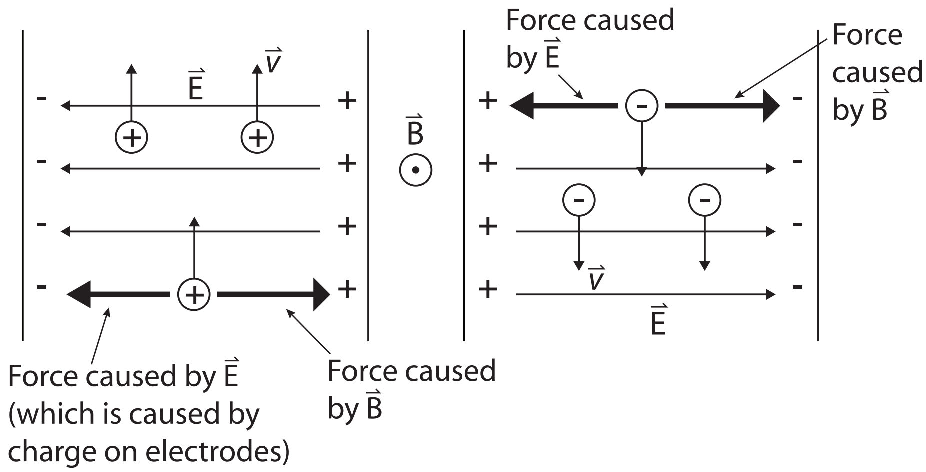

Figure 1: Figure for 1.1C. Opposite polarity voltages between holes and electrons (Image by MIT OpenCourseWare.)

A

By Gauss' Law:

$$\oint_{S_R} \epsilon_0 \vec{E} \cdot d\vec{a} = \int_{V_R} \rho dV; \quad S_r = \text{sphere with radius r}$$

As shown in class, symmetry ensures $\vec{E}$ has only radial compoent: $\vec{E} = E_r \hat{i}_r$

<u>LHS of Gauss' Law</u>:

$$\oint_{S_R} \epsilon_0 \vec{E} \cdot d\vec{a} = \int_0^{2\pi} \int_0^{\pi} \epsilon_0 \left(E_r \hat{i}_r\right) \cdot \underbrace{\left(r^2 \sin\theta d\theta d\phi \hat{i}_r\right)}_{d\vec{a} \text{ in spherical coord.}}$$

$$= \underbrace{4\pi r^2}_{\text{surface area of sphere of radius r}} E_r \epsilon_0$$

<u>RHS of Gauss' Law</u>:

<u>For $r < b$</u>:

$$\int_{V_R} \rho dV = \int_0^r \int_0^{2\pi} \int_0^{\pi} \frac{\rho r}{b} \underbrace{r^2 \sin\theta d\theta d\phi dr}_{\text{diff. vol. element}}$$

$$= \underbrace{\frac{4}{4}\frac{\pi r^4}{b}}_{\substack{\text{vol of}\\ \text{sphere}}} \rho_b = \frac{\pi r^4 \rho_b}{b}$$

<u>For $r > b$ & $r < a$ ($b < r < a$):</u>

$$\int_{V_R} \rho dV = \int_0^b \int_0^{2\pi} \int_0^{\pi} \frac{\rho_b r}{b} r^2 \sin\theta d\theta d\phi dr + \int_b^r \int_0^{2\pi} \int_0^{\pi} \rho_a r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{\cancel{4}\pi\rho_b b^3}{\cancel{4}} + \frac{4\pi\rho_a(r^3 - b^3)}{3}$$

$$= \pi\rho_b b^3 + \frac{4}{3}\pi\rho_a(r^3 - b^3) \quad b < r < a$$

B

Equating LHS and RHS

$$4\pi r^2 E_r \epsilon_0 = \begin{cases} \frac{\pi r^4}{b}\rho_b; & r < b \\ \pi\rho_b b^3 + \frac{4\pi\rho_a(a^3 - b^3)}{3}; & b < r < a \end{cases}$$

$$\boxed{E_r = \begin{cases} \frac{r^2\rho_b}{4\epsilon_0 b}; & r < b \\ \frac{b^3\rho_b}{4\epsilon_0 r^2} + \frac{\rho_a(r^3 - b^3)}{3\epsilon_0 r^2}; & b < r < a \end{cases}}$$

C

Again: $\hat{n} \cdot (\epsilon_0 E^a - \epsilon_0 E^b) = \sigma_s$

$\vec{E}(r = a^+) = 0$

$$E_r(r = a_-) = \frac{\rho_b b^3}{4\epsilon_0 a^2} + \frac{\rho_a(a^3 - b^3)}{3\epsilon_0 a^2} \leftarrow \text{ by part (a)}$$

$\sigma_s = \hat{i}_r \cdot \left(-\epsilon_0 \vec{E}(r = a^-)\right)$, so:

$$\sigma_s = -\left[\frac{\rho_b b^3}{4\epsilon_0 a^2} + \frac{\rho_a(a^3 - b^3)}{3\epsilon_0 a^2}\right]$$

D

$r < b$	$Q_b = \pi b^3 \rho_b$	$Q_\sigma(r = a) = \sigma_s 4\pi a^2 = -4\pi a^2 \left[\frac{\rho_b b^3}{4\epsilon_0 a^2} + \frac{\rho_a(a^3 - b^3)}{3\epsilon_0 a^2}\right]$
$b < r < a$	$Q_a = \frac{4}{3}\pi(a^3 - b^3)\rho_a$	$Q_\sigma = Q_b + Q_a + Q_\sigma = 0$

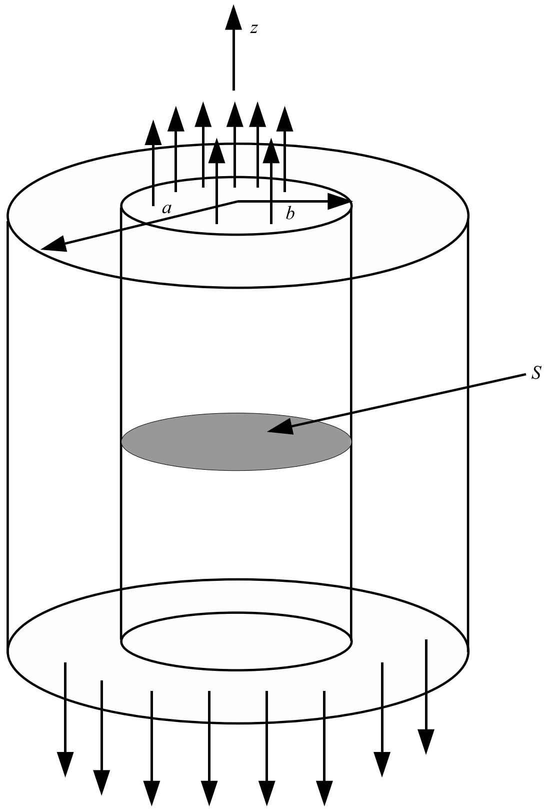

Figure 2: A diagram of a wire carrying a non-uniform current density and the return current at $r = a$ (Image by MIT OpenCourseWare).

Problem 1.3

A

We are told current in $+z$ direction inside cylinder $r < b$

Current going through cylinder:

$$= I_{total} = \int_S \vec{J} \cdot d\vec{a} = \int_0^b \int_0^{2\pi} \underbrace{\left(\frac{J_0 r}{b}\hat{i_z}\right)}_{\vec{J}} \cdot \underbrace{\left(r d\phi dr \hat{i_z}\right)}_{d\vec{a}} = \frac{J_0 2\pi b^2}{3}$$

$$|\vec{K}| = \frac{\text{Total current in sheet}}{\text{length of sheet (ie, circumference of circle of radius a)}}$$

Thus, $\vec{K}$'s units are $\frac{\text{Amps}}{\text{m}}$, whereas $\vec{J}$'s units are $\frac{\text{Amps}}{\text{m}^2}$

$$|\vec{K}| = \frac{\frac{2}{3}J_0\pi b^2}{2\pi a} = \frac{J_0 b^2}{3a}$$

$$\boxed{\vec{K} = -\frac{J_0 b^2}{3a}\hat{i_z}}$$

B

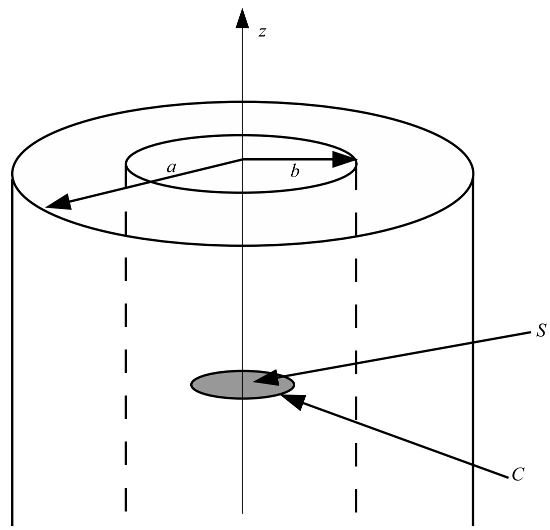

Figure 3: A diagram of the wire with a circle C centered on the z-axis with minimum surface S (Image by MIT OpenCourseWare).

Ampere's Law

$$\oint_C \vec{H} \cdot d\vec{s} = \int_S \vec{J} \cdot d\vec{a} + \underbrace{\frac{d}{dt}\int_r \epsilon_0 \vec{E} \cdot d\vec{a}}_{\text{no } \vec{E} \text{ field, term is 0}}$$

Choose C as a circle and S as the minimum surface that circle bounds.

Now solve LHS of Ampere's Law

$$\oint_C \vec{H} \cdot d\vec{s} = \int_0^{2\pi} \underbrace{(H_\phi \hat{i_\phi})}_{\vec{H}} \cdot \underbrace{(r d\phi)\hat{i_\phi}}_{d\vec{s}} = 2\pi r H_\phi$$

We assumed $H_z = H_r = 0$. This follows from the symmetry of the problem. $H_r = 0$ because $\oint_S \mu_0 \vec{H} \cdot d\vec{a} = 0$. In particular choose S as shown in Figure 4a.

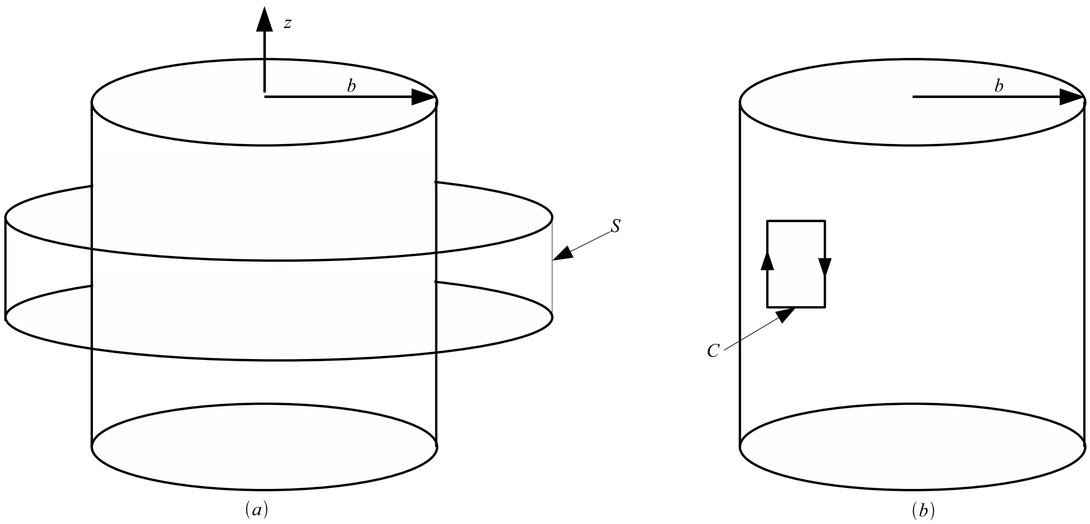

Figure 4: A diagram of the current carrying wire of radius b with the choice for S as well as a diagram of the wire with the choice of contour C (Image by MIT OpenCourseWare).

H_z is more difficult to see. It is discussed in Haus & Melcher. The basic idea is to use the contour, C (depicted in Figure 4b), to show that if $H_z \neq 0$ it would have to be nonzero even at ∞, which is not possible without sources at ∞.

Now for RHS of Ampere's Law:

$\underline{r < b}$

$$\int_S \vec{J} \cdot d\vec{a} = \int_0^{2\pi} \int_0^r \underbrace{\left(\frac{J_0 r'}{b} \hat{i_z} \right)}_{\vec{J}} \cdot \underbrace{\left(r' dr' d\phi \hat{i_z} \right)}_{d\vec{a}}$$

$$= \frac{2 J_0 r^3 \pi}{3b}$$

$\underline{a > r > b}$

$$\int_S \vec{J} \cdot d\vec{a} = \int_0^{2\pi} \int_0^b \left(\frac{J_0 r'}{b} \hat{i_z} \right) \cdot \left(r' dr' d\phi \hat{i_z} \right) + \underbrace{\int_0^{2\pi} \int_b^r \left(0 \cdot \hat{i_z} \right) \cdot \left(r' dr' d\phi \hat{i_z} \right)}_{0}$$

$$= \frac{2}{3} J_0 b^2 \pi$$

Equating LHS & RHS:

$$2\pi r H_\phi = \begin{cases} \frac{2}{3b} J_0 r^3 \pi \; ; & r < b \\ \frac{2}{3} J_0 b^2 \pi \; ; & a > r > b \\ 0 \; ; & r > a \end{cases}$$

$$\vec{H} = \begin{cases} \frac{J_0 r^2}{3b}\hat{i_\phi}\ ; & r < b \\ \frac{J_0 b^2}{3r}\hat{i_\phi}\ ; & a > r > b \\ 0\ ; & r > a \end{cases}$$

Problem 1.4

A

We can simply add the fields of the two point charges. Start with the field of a point charge q at the origin and let S_R be the sphere of radius R centered at the origin. By Gauss:

$$\oint_{S_R} \varepsilon_0 \overrightarrow{E} \cdot d\overrightarrow{a} = \int_V \rho dV$$

In this case $\rho = \delta(\overrightarrow{r})q$, so RHS is

$$\int \rho dV = \int\int\int \delta(\overrightarrow{r}) q dx dy dz = q$$

LHS is

$$\oint_{S_R} \varepsilon_0 \overrightarrow{E_r} \cdot d\overrightarrow{a} = (\varepsilon_0 \underbrace{E_r}_{\text{symmetry}})(\text{surface area of } S_r)$$

$$= 4\pi r^2 \varepsilon_0 E_r$$

Equate LHS and RHS

$$4\pi r^2 \varepsilon_0 E_r = q$$

$$\overrightarrow{E} = \frac{q}{4\pi r^2 \varepsilon_0}\hat{i}_r$$

Convert to cartesian: Any point is given by

$$\overrightarrow{r} = x(r,\theta,\phi)\hat{i}_x + y(r,\theta,\phi)\hat{i}_y + z(r,\theta,\phi)\hat{i}_z$$

By spherical coordinates

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\overrightarrow{r} = r\sin\theta\cos\phi\hat{i}_x + r\sin\theta\sin\phi\hat{i}_y + r\cos\theta\hat{i}_z$$

$\hat{i}_r \parallel$ line formed by varying r and fixing ϕ and θ

$$\bar{r} = r\bar{i_r}$$

Thus,

$$\bar{i_r} = \sin\theta\cos\phi\hat{i_x} + \sin\theta\sin\phi\hat{i}_y + \cos\theta\hat{i}_z$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i_x} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{i_y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{i_z}$$

so,

$$\overrightarrow{E} = \frac{q}{4\pi\varepsilon_0(x^2+y^2+z^2)}\hat{i}_r$$

$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 + \overrightarrow{E}_2$$

$\overrightarrow{E}_1$ is just $\overrightarrow{E}$ with $y \to y - \frac{d}{2}$. $\overrightarrow{E}_2$ is just $\overrightarrow{E}$ with $y \to y + \frac{d}{2}$. Problem has $y = 0$

(i)

$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 = \left[\frac{x}{\sqrt{x^2+\frac{d^2}{4}+z^2}}\hat{i}_x + \frac{z}{\sqrt{x^2+\frac{d^2}{4}+z^2}}\hat{i}_z - \frac{\frac{d}{2}}{\sqrt{x^2+\frac{d^2}{4}+z^2}}\hat{i}_y\right]\cdot\left[\frac{q}{4\pi\varepsilon_0(x^2+\frac{d^2}{4}+z^2)}\right]$$

$$= \frac{q}{4\pi\varepsilon_0(x^2+\frac{d^2}{4}+z^2)^{\frac{3}{2}}}\left[x\hat{i}_x - \frac{d}{2}\hat{i}_y + z\hat{i}_z\right]$$

(ii)

$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 + \overrightarrow{E}_2$$

$$= q\frac{x\hat{i}_x + z\hat{i}_z}{2\pi\varepsilon_0(x^2+\left(\frac{d}{2}\right)^2+z^2)^{\frac{3}{2}}}$$

(iii)

$$\overrightarrow{E}_{\text{total}} = \overrightarrow{E}_1 + \overrightarrow{E}_2$$

$$= \frac{-dq\hat{i}_y}{4\pi\varepsilon_0\left(x^2+\frac{d^2}{4}+z^2\right)^{\frac{3}{2}}}$$

B

$\overrightarrow{F} = q_1\overrightarrow{E}$ $\quad$ $\overrightarrow{E}$ doesn't include field of q

(i)

$\overrightarrow{F} = 0$, by Newton's third law a body cannot exert a net force on itself.

(ii)

$$\overrightarrow{F} = q_1\overrightarrow{E} = q\overrightarrow{E}_2(x=0, y=\frac{d}{2}, z=0)$$

$$= \frac{q^2\bar{i}_y}{4\pi\varepsilon_0(d^2)} = \frac{q^2\bar{i}_y}{4\pi\varepsilon_0 d^2}$$

(iii)

$$\overrightarrow{F} = -\frac{q^2\bar{i}_y}{4\pi\varepsilon_0 d^2}$$