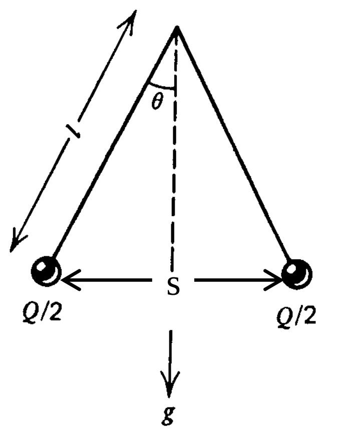
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

Additional Problems – Spring 2009

Problem 1.1 – Coulomb-Lorentz Force Law

i) An electroscope measures charge by the angular deflection of two identical conducting balls suspended by an essentially weightless insulating string of length l. Each ball has mass M in the gravity field g and when charged can be considered a point charge.

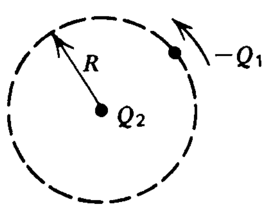


A total charge Q is deposited on the two identical balls of the electroscope when they are touching. The balls then repel each other and the string is at an angle θ from the normal which obeys a relation of the form

$$\tan\theta \sin^2\theta = constant$$

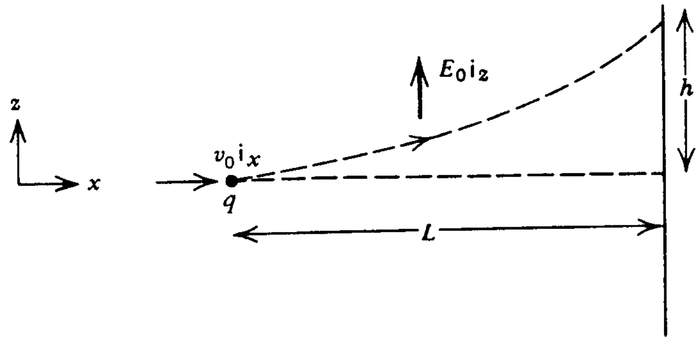
a) What is the constant?

ii) A point charge $-Q_1$ of mass m travels in a circular orbit of radius R about a stationary charge of opposite sign Q_2.

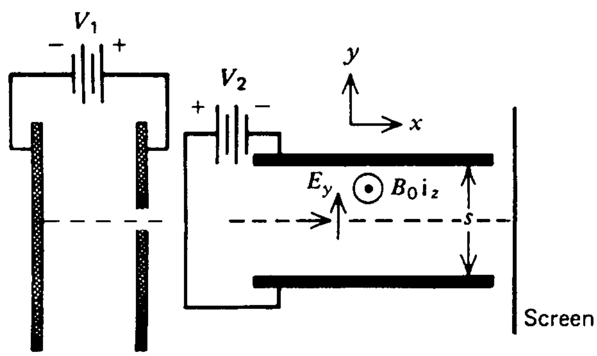


a) What is the equilibrium angular speed of the charge $-Q_1$?
b) This problem describes Bohr's one electron model of the atom if the charge $-Q_1$ is that of an electron and $Q_2=Ze$ is a nuclear charge, where Z is the number of protons. According to the postulates of quantum mechanics the angular momentum L of the electron must be quantized, $L = mvR = nh/2\pi$, n = 1, 2, 3,… where $h = 6.63\times10^{-34}$ joule-sec is Planck's constant. What are the allowed values of R?
c) For the hydrogen atom (Z=1), what is the radius of the smallest allowed orbit and what is the electron orbital velocity?

iii) A charge q of mass m with initial velocity $\mathrm{v} = v_0 \bar{\mathrm{i}}_x$ is injected at $x=0$ into a region of uniform electric field $\bar{E} = E_0 \bar{i}_z$. A screen is placed at the position $x=L$. At what height h does the charge hit the screen? Neglect gravity.



iv) The charge to mass ratio of an electron *e/m* was first measured by Sir J. J. Thomson in 1897 by the cathode-ray tube device shown. Electrons emitted by the cathode pass through a slit in the anode into a region with crossed electric and magnetic fields, both being perpendicular to the electrons velocity. The end screen of the tube is coated with a fluorescent material that produces a bright spot where the electron beam impacts.



a) What is the velocity of the electrons when passing through the slit if their initial cathode velocity is v_0?

b) The electric field $\bar{E} = (V_2 / s)\bar{i}_y$ and magnetic field $\bar{B} = B_0 \bar{i}_z$ are adjusted so that the vertical deflection of the beam is zero. What is the initial electron velocity v_0 in terms of $V_1, V_2, s, B_0, e,$ and m? (Neglect gravity.)

c) The voltage V_2 is now set to zero. What is the radius R of the electrons motion about the magnetic field in terms of $V_1, B_0, v_0, e,$ and m?

d) With $V_2 = 0$, what is *e/m* in terms of V_1, B_0, v_0, and R?

v) A point charge q with mass m and velocity $\bar{v}$ moves in a vacuum through a magnetic field $\bar{H}$. Newton's law for this charge is:

$$m\frac{d\vec{v}}{dt} = q\vec{v} \times \mu_o \vec{H}$$

A uniform magnetic field in the *z* direction is imposed

$$\vec{H} = H_o \vec{i}_z$$

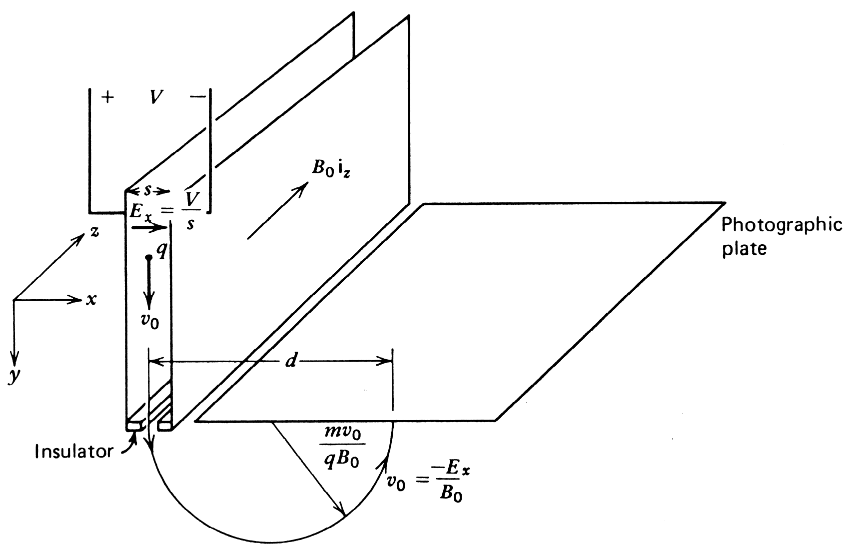
Solve Newton's law for the three velocity components v_x, v_y, and v_z for initial conditions

$$\vec{v}(t=0) = v_{xo}\vec{i}_x + v_{yo}\vec{i}_y + v_{zo}\vec{i}_z$$

Note that the velocity magnitude is constant so that a circular motion results in the *x*-*y* plane.

a) What is the radius of the circle?

b)



The mass spectrograph uses the circular motion to determine the masses of ions and to measure the relative proportions of isotopes. Charges enter between parallel plate electrodes with a *y* directed velocity distribution. Gravitational effects are negligible compared to the electric/magnetic forces since the ions have so little mass.

To pick out those charges with a particular magnitude of velocity, perpendicular electric and magnetic fields are imposed so that the net force on a charge is

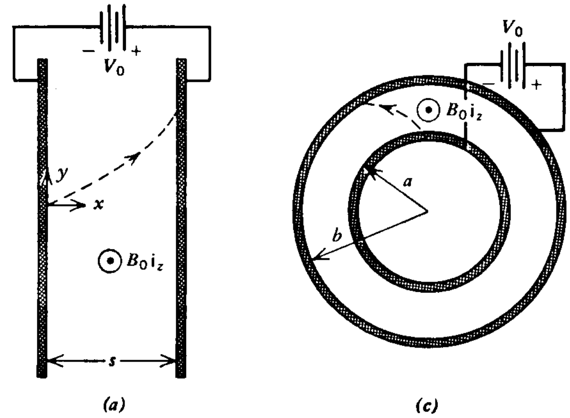
$$\vec{f} = q(\vec{E} + \vec{v} \times \mu_o \vec{H})$$

For charges to pass through the narrow slit at the end of the channel, they must not be deflected by the fields which require this force to be zero. For a selected velocity $v_y = v_o$, what is the required applied electric field and thus the necessary voltage *V* for a given magnetic field $H = H_o \vec{i}_z$ so that an ion will make it through this slit undeflected? The plate electrodes have a spacing, *s*, as shown in the figure.

c) Note that the charge circular path diameter *d* depends on the ion mass, and so can be used to detect different isotopes that have the same number of protons but a different number of neutrons. The isotopes thus have the same charge but different masses. Typically $V = -100$ volts across a *s* =1 cm gap with a magnetic field of μ_o H=1 tesla. The mass of a proton and neutron are each about m=1.67 x 10^{-27} kg. Consider the three isotopes of magnesium $12Mg^{24}$, $12Mg^{25}$, and $12Mg^{26}$, each deficient of one electron. At what positions *d* will each isotope hit the photographic plate?

Problem 1.2

A magnetron is essentially a parallel plate capacitor stressed by constant voltage V_o where electrons of charge -e are emitted at *x=0, y=0* with zero initial velocity. A transverse magnetic field $B_o \vec{i}_z$ is applied. Neglect the electric and magnetic fields due to the electrons in comparison to the applied field.



(a) What is the velocity and displacement of an electron, injected from the cathode with zero initial velocity at *t=0*?
(b) What value of magnetic field will just prevent the electrons from reaching the other electrode? This is the cut-off magnetic field.
(c) A magnetron is built with coaxial electrodes where electrons are injected from $r = a$, $\phi = 0$ with zero initial velocity. Using the relations

$$\boldsymbol{i}_r = \cos\phi \boldsymbol{i}_x + \sin\phi \boldsymbol{i}_y$$

$$\boldsymbol{i}_\phi = -\sin\phi \boldsymbol{i}_x + \cos\phi \boldsymbol{i}_y$$

show that

$$\frac{d\boldsymbol{i}_r}{dt} = \boldsymbol{i}_\phi \frac{d\phi}{dt} = \frac{v_\phi}{r} \boldsymbol{i}_\phi$$

$$\frac{d\boldsymbol{i}_\phi}{dt} = -\boldsymbol{i}_r \frac{d\phi}{dt} = -\frac{v_\phi}{r}\boldsymbol{i}_r$$

What is the acceleration of a charge with velocity

$$\boldsymbol{v} = v_r \boldsymbol{i}_r + v_\phi \boldsymbol{i}_\phi \,?$$

(d) Find the velocity of the electron as a function of radial position.

Hint:

$$\frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r \frac{dv_r}{dr} = \frac{d}{dr}\left(\frac{1}{2}v_r^2\right)$$

$$\frac{dv_\phi}{dt} = \frac{dv_\phi}{dr}\frac{dr}{dt} = v_r \frac{dv_\phi}{dr}$$

(e) What is the cutoff magnetic field? Check your answer with (b) in the limit $b = a + s$ where $s \ll a$.

Problem 1.3

In a spherically symmetric configuration, the region $r < b$ has the non-uniform charge density $\rho_b r/b$ and is surrounded by a region $b < r < a$ having the uniform charge density ρ_a. At $r = b$ there is no surface charge density, while at $r = a$ there is a perfectly conducting sheet with surface charge density that assures $\boldsymbol{E} = 0$ for $r > a$.

(a) What is the total charge in the regions $0 < r < b$ *and* $b < r < a$?

(b) Determine $\boldsymbol{E}$ in the two regions $r < b$ and $b < r < a$.

(c) What is the surface charge density at $r = a$?

(d) What is the total charge in the system for $0 < r \le a$.

Problem 1.4

In polar coordinates, a non-uniform current density $J_0 r / b\bar{\boldsymbol{i}}_z$ exists over the cross-section of a wire having a radius b. The total current in the wire is returned in the -z direction as a uniform surface current at the radius $r = a$ where $a > b$.

(a) What is the surface current density at $r = a$?

(b) Find the magnetic field in the regions $0 < r < b$, $b < r < a$, and $r > a$.

Problem 1.5

Write a brief paragraph of what you saw and what you learned from viewing each of the assigned video demonstrations.

Demos: 1.3.1 and 1.5.1, and 11.7.1.

Problem 2.1

The superposition integral for the electric scalar potential is

$$\Phi(\bar{r}) = \int_{V'} \frac{\rho(\bar{r}')dV'}{4\pi\varepsilon_o |\bar{r} - \bar{r}'|} \tag{1}$$

The electric field is related to the potential as

$$\bar{E}(\bar{r}) = -\nabla\Phi(\bar{r}) \tag{2}$$

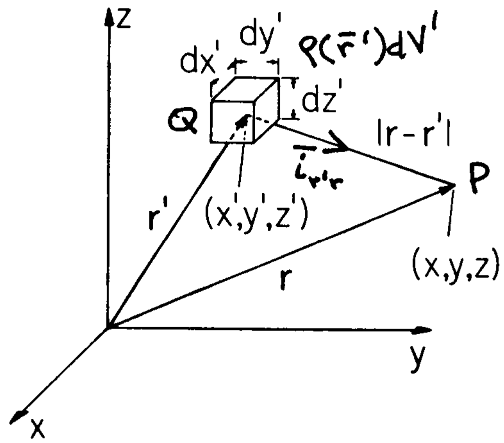


Figure 4.5.1 An elementary volume of charge at **r′** gives rise to a potential at the observer position **r**.

Figure 4.5.1 from *Electromagnetic Fields and Energy* by Hermann A. Haus and James R. Melcher. Used with permission.

The vector distance between a source point at Q and a field point at P is:

$$\bar{r} - \bar{r}' = (x - x')\bar{i}_x + (y - y')\bar{i}_y + (z - z')\bar{i}_z \tag{3}$$

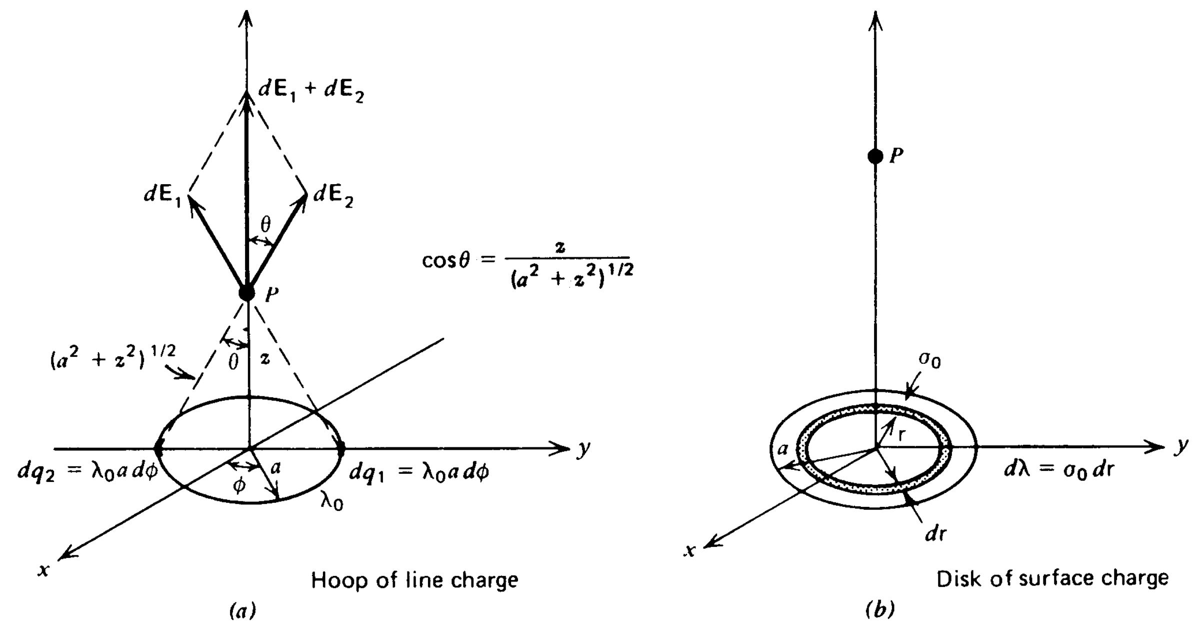
(a) By differentiating $|\vec{r} - \vec{r}'|$ in Cartesian coordinates with respect to the unprimed coordinates at P show that

$$\nabla\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) = \frac{-(\bar{r} - \bar{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{-\bar{i}_{r'r}}{|\vec{r} - \vec{r}'|^2} \tag{4}$$

where $\bar{i}_{r'r}$ is the <u>unit</u> vector pointing from Q to P.

(b) Using the results of (a) show that

$$\bar{E}(\bar{r}) = -\nabla \Phi(\bar{r}) = -\int_{V'} \frac{\rho(\bar{r}')}{4\pi\varepsilon_o} \nabla\left(\frac{1}{|\bar{r}-\bar{r}'|}\right) dV' = \int_{V'} \frac{\rho(\bar{r}')\bar{i}_{r'r}}{4\pi\varepsilon_o |\bar{r}-\bar{r}'|^2} dV' \qquad (5)$$



(c) A circular hoop of line charge λ_0 coulombs/meter with radius a is centered about the origin in the $z=0$ plane. Find the electric scalar potential along the z-axis for $z<0$ and $z>0$ using Eq. (1) with $\rho(r')dV' = \lambda_o a d\phi$. Then find the electric field magnitude and direction using symmetry and $\bar{E} = -\nabla\Phi$. Verify that using the last integral in Eq. (5) gives the same electric field. What do the electric scalar potential and electric field approach as $z \to \infty$ and how do these results relate to the potential and electric field of a point charge?

(d) Use the results of (c) to find the electric scalar potential and electric field along the z axis for a uniformly surface charged circular disk of radius a with uniform surface charge density σ_0 coulombs/m^2. Consider $z>0$ and $z<0$.

(e) What do the electric scalar potential and electric field approach as $z \to \infty$ and how do these results relate to the potential and electric field of a point charge?

(f) What do the potential and electric field approach as the disk gets very large so that $a \to \infty$

(g) Consider the case where the line charge density of (a) is a function of angle ϕ, $\lambda(\phi)=\lambda_0 \sin \phi$. What is the electric scalar potential along the z-axis? Can you use Eq. (1) to find the electric field along the z-axis?

(h) Use Eq. (5) to find the electric field along the z-axis.
Find $\lim_{z\to\infty} \bar{E}(r=0,z)$.

(i) Now consider that the surface charge density in (b) is a function of ϕ, $\sigma(\phi) = \sigma_0 \sin\phi$. What is the electric scalar potential along the z-axis? Can you use Eq. (1) to find the electric field along the z-axis?

(j) Use the results of (h) to find the electric field along the z-axis. Find $\lim_{z\to\infty} \bar{E}(r=0,z)$.

Problem 2.2

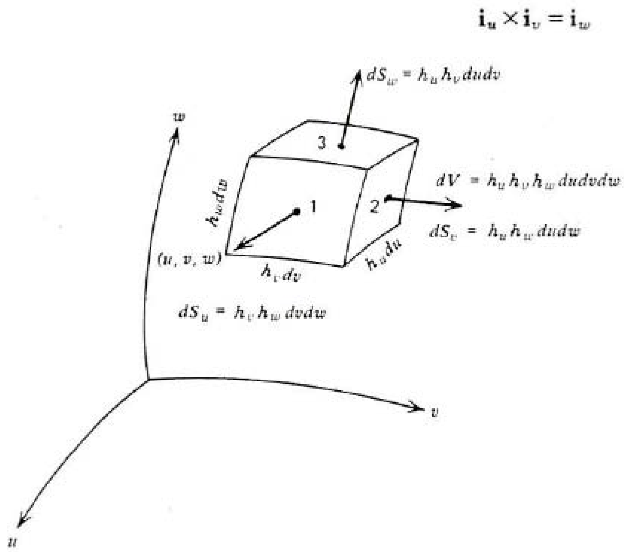
The curl and divergence operations have a simple relationship that will be used throughout the subject.

(a) One might be tempted to apply the divergence theorem to the surface integral in Stokes' theorem. However, the divergence theorem requires a closed surface while Stokes' theorem is true in general for an open surface. Stokes' theorem for a closed surface requires the contour to shrink to zero giving a zero result for the line integral. Use the divergence theorem applied to the closed surface with vector $\nabla\times\vec{A}$ to prove that $\nabla\bullet(\nabla\times\vec{A})=0$.

(b) Verify (a) by direct computation in Cartesian and spherical coordinates.

Problem 2.3

A general right-handed orthogonal curvilinear coordinate system is described by variables (u, v, w), where



Since the incremental coordinate quantities *du*, *dv*, and *dw* do not necessarily have units of length, the differential length elements must be multiplied by coefficients that generally are a function of *u*, *v*, and *w*:

$$dL_u = h_u d_u, \quad dL_v = h_v d_v, \quad dL_w = h_w d_w$$

(a) What are the *h* coefficients for the Cartesian, cylindrical, and spherical coordinate systems?
(b) What is the gradient function of any function *f(u,v,w)*?
(c) What is the area of each surface and the volume of a differential size volume element in the (*u*, *v*, *w*) space?
(d) What are the curl and divergence of the vector

$$\mathbf{A} = A_u \mathbf{i_u} + A_v \mathbf{i_v} + A_w \mathbf{i_w}?$$

(e) What is the scalar Laplacian $\nabla^2 f = \nabla \cdot (\nabla f)$?

(f) Check your results of (b)-(e) for the three basic coordinate systems.

<u>Problem 2.4</u>
Write a brief paragraph of what you saw and what you learned from viewing each of the assigned video demonstrations.

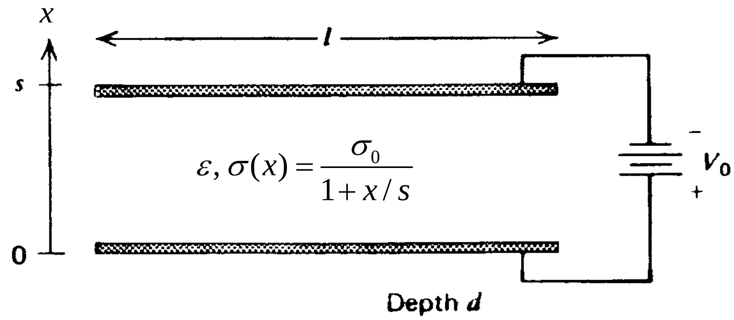
Demos: 4.7.1 and 10.2.1

<u>Problem 3.1</u>
Write a brief paragraph of what you saw and what you learned from viewing each of the assigned video demonstrations.

Demos: 8.2.1, 8.2.2, and 8.4.1

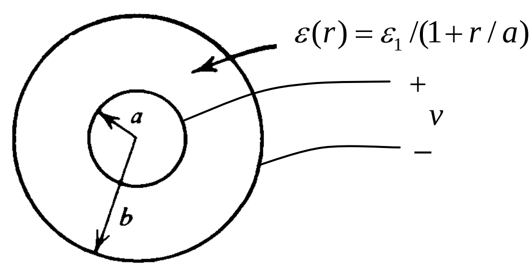
Problem 4.1

A pair of parallel plate electrodes at voltage difference V_0 enclose an Ohmic material whose conductivity varies with position as $\sigma(x) = \dfrac{\sigma_0}{1+x/s}$. The permittivity ε of the material is a constant.



a) Find the electric field $E_x(x)$ and the resistance between the electrodes.
b) What are the volume and surface charge densities?
c) What is the total volume charge in the system and how is it related to the total surface charge on the electrodes?

Problem 4.2



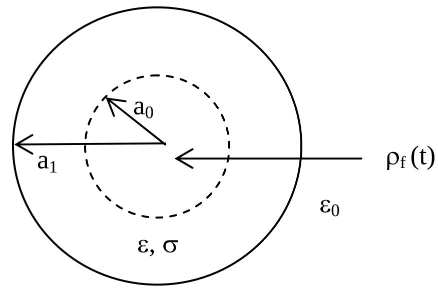
Concentric spherical electrodes with respective radii *a* and *b* enclose a material whose permittivity varies with radius as $\varepsilon(r) = \varepsilon_1/(1+r/a)$. A voltage *v* is applied across the spherical electrodes. There is no volume charge in the dielectric.

a) What are the electric field and potential distributions for $a<r<b$?
b) What are the surface charge densities at $r=a$ and $r=b$?
c) What is the capacitance?

Problem 4.3

An infinitely long cylinder of radius a_1, permittivity ε, and conductivity σ is nonuniformly charged at $t=0$:

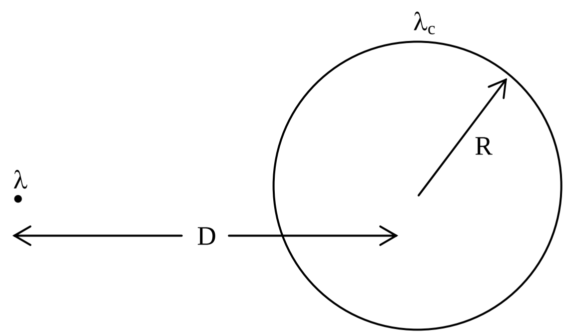
$$\rho_f(t=0)=\begin{cases}\frac{\rho_o r^2}{a_o^{\,2}} & 0<r<a_o\\ 0 & r>a_o\end{cases}$$



What is the time dependence of the electric field everywhere and the free surface charge density at $r=a_1$ as a function of time? At time t=0 the surface charge at r=a, is zero.

Problem 4.4

An infinity long line charge λ is a distance D from the center of a conducting cylinder of radius R that carries a total charge per unit length λ_c. What is the force per unit length on the cylinder?



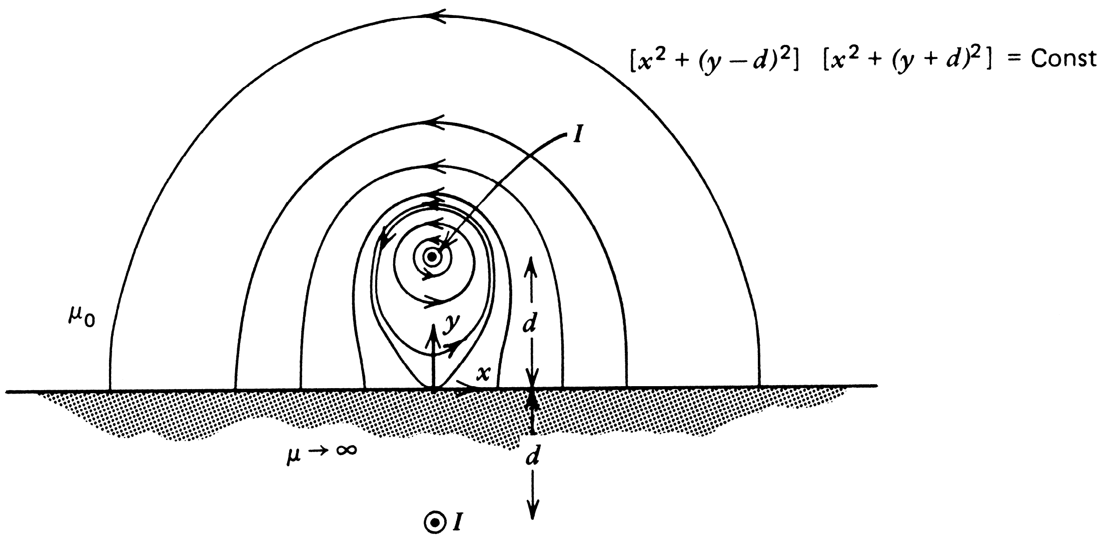
(**Hint**: Where can another image charge be placed with the cylinder remaining an equipotential surface?)

Problem 4.5

Write a brief paragraph of what you saw and what you learned from viewing each of the assigned video demonstrations.

Demos: 1.4.1, 1.6.1, 6.6.1, and 9.4.1.

Problem 5.1



A line current I of infinite extent in the *z* direction is at a distance d above an infinitely permeable material as shown above.

a. What is the boundary condition on the magnetic field at $y=0$?
b. Use the method of images to satisfy the boundary condition of (a) and find the magnetic vector potential for $y>0$.
c. What is the magnetic field for $y>0$?
d. What is the force per unit length on the line current at $y=d$?