6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn
Quiz 2: Solution

1) (a) at $0 < x < d \quad \bar{H} = H_y \bar{i}_y$

at $x < 0 \;,\; \mu \to \infty \quad \bar{H} = 0$

at $d < x \;,\; \sigma \to \infty \quad \bar{H} = 0$

(b) at $x = 0 \quad H_y\left(0^+\right) - H_y\left(0^-\right) = K_z\left(0\right)$

$$H_y\left(0^+\right) = K_0 \cos \omega t$$

at $x = d \quad B_x\left(d^+\right) = B_x\left(d^-\right) = 0$

$$-H_y\left(d^-\right) = K_z\left(d\right)$$

(c) at $0 < x < d \quad \bar{J} = J_z \bar{i}_z \;,\; \bar{E} = E_z \bar{i}_z$

at $x = d \quad E_z\left(d^+\right) = E_z\left(d^-\right) = 0$

(d) $\nabla^2 \bar{H} - \omega_p^{\,2} \varepsilon_0 \mu_0 \bar{H} = 0 \;;\quad \nabla \times \bar{H} = \bar{J} \Rightarrow J_z = \dfrac{\partial H_y}{\partial x} \;;\; \dfrac{\partial \bar{J}}{\partial t} = \omega_p^{\,2} \varepsilon_0 \bar{E} \Rightarrow E_z = \dfrac{1}{\omega_p^{\,2} \varepsilon_0} \dfrac{\partial J_z}{\partial t}$

$$\frac{d^2 H_y}{dx^2} - \frac{\omega_p^{\,2}}{c^2} H_y = 0 \qquad , \; c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$H_y = A_1 e^{\omega_p x/c} + A_2 e^{-\omega_p x/c}$$

$$J_z = B_1 e^{\omega_p x/c} + B_2 e^{-\omega_p x/c} = \frac{\omega_p}{c}\left(A_1 e^{\omega_p x/c} - A_2 e^{\omega_p x/c}\right)$$

$$E_z = C_1 e^{\omega_p x/c} + C_2 e^{-\omega_p x/c} = \frac{1}{\omega_p c \varepsilon_0}\left(\frac{\partial A_1}{\partial t} e^{\omega_p x/c} - \frac{\partial A_2}{\partial t} e^{-\omega_p x/c}\right)$$

(e) $\left.\begin{array}{l} H_y(x=0) = K_0 \cos\omega t \\ E_z(x=d) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A_1 = \dfrac{e^{-2\omega_p d/c}}{1+e^{-2\omega_p d/c}} K_0 \cos\omega t \\ \\ A_2 = \dfrac{1}{1+e^{-2\omega_p d/c}} K_0 \cos\omega t \end{array}$

$$H_y = \frac{K_0 \cos\omega t}{1+e^{-2\omega_p d/c}}\left(e^{-2\omega_p d/c} e^{\omega_p x/c} + e^{-\omega_p x/c}\right) = \frac{K_0 \cos\omega t \cosh\left(\omega_p (x-d)/c\right)}{\cosh\left(\omega_p (d/c)\right)}$$

$$J_z = \frac{\omega_p}{c} \frac{K_0 \cos\omega t}{1+e^{-2\omega_p d/c}}\left(e^{-2\omega_p d/c} e^{\omega_p x/c} - e^{-\omega_p x/c}\right) = \frac{\omega_p}{c} \frac{K_0 \cos\left(\omega_p d/c\right) \sinh\left(\omega_p (x-d)/c\right)}{\cosh\left(\omega_p d/c\right)}$$

$$E_z = \frac{-\omega K_0 \sin\omega t}{\omega_p c \varepsilon_0 \left(1+e^{-2\omega_p d/c}\right)} \left(e^{-2\omega_p d/c} e^{\omega_p x/c} - e^{-\omega_p x/c}\right) = -\frac{K_0 \omega}{\omega_p \varepsilon_0 c} \frac{\sin\left(\omega_p d/c\right) \sinh\left(\omega_p (x-d)/c\right)}{\cosh\left(\omega_p (d/c)\right)}$$

(f) $\bar{K}(x=d) = \bar{i}_z \dfrac{-K_0 \cos\omega t}{1+e^{-2\omega_p d/c}} 2e^{-\omega_p d/c} = \bar{i}_z \dfrac{-K_0 \cos\omega t}{\cosh\left(\omega_p d/c\right)}$

(g) $\bar{f} = \left.\dfrac{\bar{K} \times \bar{B}}{2}\right|_{x=d} = \left.\dfrac{1}{2} K_z \bar{i}_z \times \mu_0 H_y \bar{i}_y\right|_{x=d} = \left.\dfrac{\bar{i}_x}{2} \mu_0 K_z^2\right|_{x=d}$

$$= \bar{i}_x \frac{2\mu_0 K_0^2 \cos^2\omega t}{\left(1+e^{-2\omega_p d/c}\right)^2} e^{-2\omega_p d/c}$$

$$= \bar{i}_x \frac{1}{2} \frac{\mu_0 K_0^2 \cos^2\omega t}{\cosh^2\left(\omega_p d/c\right)}$$

2) (a) $\nabla \cdot \bar{B} = 0$, $\bar{B} = \mu_0 \bar{H}$, $\bar{H} = -\nabla\chi \Rightarrow \nabla^2\chi = 0, \quad r < R$

$\bar{B} = \mu_0\left(\bar{H} + \bar{M}\right)$, $\nabla \cdot \bar{H} = -\nabla \cdot \bar{M} = 0 \Rightarrow \nabla^2\chi = 0, \quad r > R$

(b) at $r = R \quad H_\theta\left(R^-\right) = H_\theta\left(R^+\right)$ (no surface current)

$$B_r\left(R^-\right) = B_r\left(R^+\right)$$

(c) $$\chi = \begin{cases} Ar\cos\theta \;, & r < R \\ \dfrac{B}{r^2}\cos\theta \;, & r > R \end{cases} \qquad \overline{H} = -\nabla\chi = -\overline{i}_r \frac{\partial\chi}{\partial r} - \overline{i}_\theta \frac{1}{r}\frac{\partial\chi}{\partial\theta}$$

$$\overline{H} = \begin{cases} -\overline{i}_r A\cos\theta + \overline{i}_\theta A\sin\theta \;, & r < R \\ 2\overline{i}_r \dfrac{B}{r^3}\cos\theta + \overline{i}_\theta \dfrac{B}{r^3}\sin\theta \;, & r > R \end{cases}$$

use boundary conditions $\quad A = \dfrac{B}{R^3}, \; -A\cos\theta = \dfrac{2B}{R^3}\cos\theta + M_0\cos\theta$

$$\Rightarrow \quad A = -\frac{M_0}{3} \qquad B = -\frac{R^3}{3}M_0$$

$$\chi = \begin{cases} -\dfrac{M_0}{3} r\cos\theta, & r < R \\ -\dfrac{M_0 R^3}{3r^2}\cos\theta, & r > R \end{cases} \qquad \overline{H} = \begin{cases} \overline{i}_r \dfrac{M_0}{3}\cos\theta - \overline{i}_\theta \dfrac{M_0}{3}\sin\theta \;, & r < R \\ -\overline{i}_r \dfrac{2M_0 R^3}{3r^3}\cos\theta - \overline{i}_\theta \dfrac{M_0 R^3}{3r^3}\sin\theta \;, & r > R \end{cases}$$

(d) $$\frac{\left|\overline{m}_e\right|\cos\theta}{4\pi r^2} = -\frac{\left|M_0 R^3\right|}{3r^2}\cos\theta \Rightarrow \overline{m}_e = -M_0 \frac{4\pi R^3}{3}\overline{i}_z$$

3) (a) $H = \dfrac{Ni}{g}$ in air gap and the magnetic material gap, $B_{\mu_0} = \dfrac{\mu_0 Ni}{g}, \; B_\mu = \dfrac{\mu Ni}{g}$

$$\Phi = \mu_0 Hwx + \mu Hw(D - x) = \frac{Niw}{g}\left(\mu_0 x + \mu(D - x)\right)$$

$$\lambda = N\Phi = \left(\mu_0 x + \mu(D - x)\right)\frac{N^2 iw}{g}$$

(b) $dW' = \lambda di + f_m dx$

$$W' = \int \lambda di + \int f_m dx = \left(\mu_0 x + \mu(D - x)\right)\frac{N^2 w}{2g} I_0^2$$

$$f_m = \left.\frac{dW'}{dx}\right|_i = \frac{-(\mu-\mu_0)N^2 w}{2g} I_0^{\,2}$$

(c) $$v = \left.\frac{d\lambda}{dt}\right|_{i=I_0} = \frac{d}{dt}\Big(\mu D + (\mu_0-\mu)\,(x_0 + x'\cos\omega t)\Big)\,\frac{N^2\,I_0\,w}{g}$$

$$= (\mu-\mu_0)\,\omega\sin\omega t\,\frac{N^2\,I_0\,w}{g}\,x'$$

(d) In a cycle $\Delta W' = 0$ $\quad \oint \lambda di + \oint f_m dx = 0$

total work $-\oint f_m dx = +\oint \lambda di$

$$= -\left[\int_{x_1}^{x_2} f_m\,(I_1)\,dx + \int_{x_2}^{x_1} f_m\,(I_2)\,dx\right]$$

$$= -\frac{\mu-\mu_0}{2g} N^2\,w\,(x_2 - x_1)\left(I_2^2 - I_1^2\right)$$