6.641, Electromagnetic Fields, Forces, and Motion
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Final Exam: Solution

1. a) apply charge relaxation for inner cylinder $(r < R_1)$

$$\frac{\partial}{\partial t}\rho_f + \frac{\sigma}{\varepsilon}\rho_f = 0 \;\Rightarrow\; \rho_f = \rho_0\, e^{-t/\tau} \quad , \tau = \frac{\varepsilon}{\sigma}$$

use Gauss' Law

$$\oint_S \overline{E}\cdot\overline{da} = \int \frac{\rho_f dV}{\varepsilon} \;\Rightarrow\; E_r 2\pi r d = \frac{\rho_f}{\varepsilon}\lambda r^2 d,\; E_r = \frac{\rho_f r}{2\varepsilon} = \frac{\rho_0 r}{2\varepsilon} e^{-t/\tau}$$

b) total charge within the inner cylinder is constant. Gauss' Law

for $R_1 < r < R_2$

$$\oint_S \overline{E}\cdot\overline{da} = \int \frac{\rho_f}{\varepsilon_0} dV \;\Rightarrow\; E_r 2\pi r d = \frac{\rho_0}{\varepsilon_0}\pi R_1^2 d,\quad E_r = \frac{\rho_0 R_1^2}{2 r \varepsilon_0}$$

c) $$\overline{n}\cdot\left[\varepsilon_0 \overline{E}\Big|_{R_{1+}} - \varepsilon \overline{E}\Big|_{R_{1-}}\right] = \sigma_{sf}(r = R_1)$$

$$\sigma_{sf}(r = R_1) = \varepsilon_0 E_r\Big|_{r=R_{1+}} - \varepsilon E_r\Big|_{r=R_{1-}}$$

$$= \frac{\rho_0 R_1}{2} - \frac{\rho_0 R_1}{2} e^{-t/\tau} = \frac{\rho_0 R_1}{2}\left(1 - e^{-t/\tau}\right)$$

$$\sigma_{sf}(r = R_2) = -\varepsilon_0 E_r\Big|_{r=R_{2-}} = -\frac{\rho_0 R_1^2}{2 R_2}$$

2. a) $\overline{P} = P_0 \overline{i}_z = P_0\left(\cos\theta\, i_r - \sin\theta\, \overline{i}_\theta\right)$

b) $\Phi(r = R) = 0$

$\Phi(r = R_{P+}) = \Phi(r = R_{P-})$

$\varepsilon_0 E_r(r = R_{P+}) = \varepsilon_0 E_r(r = R_{P-}) + P_0 \cos\theta$

c) $\bar{E} = -\nabla\Phi \ , \ \nabla^2\Phi = 0$

$$\Phi = \begin{cases} Ar\cos\theta \quad , \quad r < R_P \\ Br\cos\theta + \dfrac{D}{r^2}\cos\theta, \ R_P < r < R \end{cases}$$

$$\bar{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\bar{i}_r - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\bar{i}_\theta = \begin{cases} -A\cos\theta\,\bar{i}_r + A\sin\theta\,\bar{i}_\theta \ , \quad r < R_P \\ -\left(B\cos\theta - \dfrac{2D}{r^3}\cos\theta\right)\bar{i}_r + \left(B + \dfrac{D}{r^3}\right)\sin\theta\ \bar{i}_\theta \qquad R_P < r < R \end{cases}$$

$$\text{BC}: \quad BR + \frac{D}{R^2} = 0$$

$$AR_P = BR_P + \frac{D}{R_p^2}$$

$$-\left(B - \frac{2D}{R_p^3}\right) = -A + \frac{P_0}{\varepsilon_0}$$

$$A = \frac{P_0}{3\varepsilon_0}\left(1 - \left(\frac{R_P}{R}\right)^3\right), \quad B = \frac{-P_0}{3\varepsilon_0}\left(\frac{R_P}{R}\right)^3, \quad D = \frac{P_0}{3\varepsilon_0}R_P^3$$

$$\bar{E} = \begin{cases} -\dfrac{P_0}{3\varepsilon_0}\left(1 - \left(\dfrac{R_P}{R}\right)^3\right)\left[\cos\theta\,\bar{i}_r - \sin\theta\,\bar{i}_\theta\right], \quad r < R_P \\ \dfrac{P_0}{3\varepsilon_0}\left(\dfrac{R_P}{R}\right)^3\left[\cos\theta\,\bar{i}_r - \sin\theta\,\bar{i}_\theta\right] + \dfrac{P_0}{3\varepsilon_0}\dfrac{R_P^3}{r^3}\left[2\cos\theta\,\bar{i}_r + \sin\theta\,\bar{i}_\theta\right], \quad R_P < r < R \end{cases}$$

d) $\sigma_{sf}\big|_{r=R} = -\varepsilon_0 E_r\big|_{r=R} = -3P_0\left(\dfrac{R_P}{R}\right)^3\cos\theta$

3. a) $\bar{H} = \frac{I_0}{D}\,\bar{i}_z$

b) EQS system

$$\left.\begin{aligned} &\nabla \times \bar{E} = -\frac{\partial}{\partial t}\left(\mu \bar{H}\right) \\ &\nabla \times \bar{H} = \bar{J} = \sigma \bar{E} \\ &\nabla \bullet \left(\mu \bar{H}\right) = 0 \end{aligned}\right\} \Rightarrow \quad \nabla^2 \bar{H} = \mu\sigma \frac{\partial \bar{H}}{\partial t}$$

For DC Steady State $\frac{\partial}{\partial t} = 0$, $\nabla^2 \bar{H} = 0$

We just have z component H_z in the material, $\nabla^2 H_z = 0$

c) As field has no variation in the y and z directions

$$\frac{\partial^2 H_z}{\partial x^2} = 0 \Rightarrow H_z = \begin{cases} Ax + B, & 0 < x \le s \\ Cx + G, & s < x \le 2s \end{cases}$$

$$\bar{J} = \nabla \times \bar{H} \Rightarrow \bar{J} = J_y \bar{i}_y, \quad J_y = -\frac{\partial}{\partial x} H_z = \begin{cases} -A \quad , & 0 < x \le s \\ -C \quad , & s < x \le 2s \end{cases}$$

$$\bar{E} = \frac{\bar{J}}{\sigma} \Rightarrow \bar{E} = E_y \bar{i}_y, \quad E_y = \frac{J_y}{\sigma} = \begin{cases} -\dfrac{A}{\sigma_1} \quad , & 0 < x \le s \\ -\dfrac{C}{\sigma_2} \quad , & s < x \le 2s \end{cases}$$

d)

$$\left.\begin{aligned} H_z(x=0) &= \frac{I_0}{D} \\ H_z(x=2s) &= 0 \\ H_z(x=s_+) &= H_z(x=s_-) \\ E_y(x=s_+) &= E_y(x=s_-) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} & B = \frac{I_0}{D} \\ & 2sC + G = 0 \\ & As + B = Cs + G \\ & \frac{A}{\sigma_1} = \frac{C}{\sigma_2} \end{aligned}$$

e)

$$A = -\frac{I_0}{sD\left(1+\frac{\sigma_2}{\sigma_1}\right)}, \quad B = \frac{I_0}{D}, \quad C = -\frac{I_0}{sD\left(1+\frac{\sigma_1}{\sigma_2}\right)}, \quad G = +\frac{2I_0}{D\left(1+\frac{\sigma_1}{\sigma_2}\right)}$$

$$H_z = \begin{cases} -\dfrac{I_0}{sD\left(1+\frac{\sigma_2}{\sigma_1}\right)}x + \dfrac{I_0}{D}, & 0 < x \le s \\[2ex] -\dfrac{I_0}{sD\left(1+\frac{\sigma_1}{\sigma_2}\right)}x + \dfrac{2I_0}{D\left(1+\frac{\sigma_1}{\sigma_2}\right)}, & s < x \le 2s \end{cases}$$

$$J_y = \begin{cases} \dfrac{I_0}{sD\left(1+\frac{\sigma_2}{\sigma_1}\right)}, & 0 < x \le s \\[2ex] \dfrac{I_0}{sD\left(1+\frac{\sigma_1}{\sigma_2}\right)}, & s < x \le 2s \end{cases}$$

$$E_y = \begin{cases} \dfrac{I_0}{sD(\sigma_1+\sigma_2)}, & 0 < x \le s \\[2ex] \dfrac{I_0}{sD(\sigma_1+\sigma_2)}, & s < x \le 2s \end{cases}$$

f)

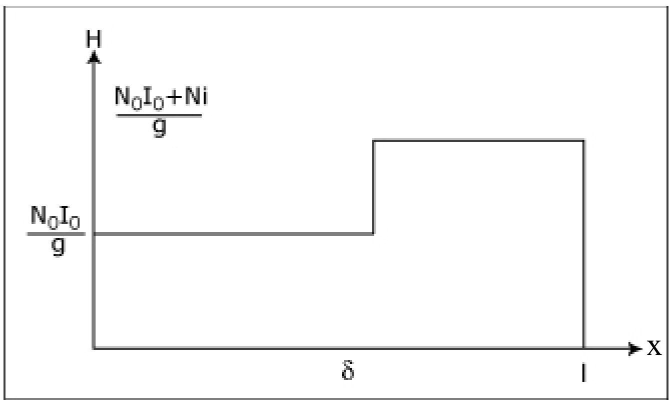
$$f_x = \oint_S T_{ij}\, n_j = -aD \; T_{xx}\Big|_{x=s} + aD \; \cancelto{0}{T_{xx}}\Big|_{x=2s} = aD\frac{\mu}{2}H_z^2\Big|_{x=s}$$

$$= \frac{aD}{2}\left[\mu\left(\frac{I_0}{D}\right)^2\left(\frac{\sigma_2}{\sigma_1+\sigma_2}\right)^2\right]$$

$$= \frac{a}{2D}\left(\frac{I_0}{\sigma_1 + \sigma_2}\right)^2 \left[\mu \sigma_2^{\ 2}\right]$$

$$T_{xx} = \frac{\mu}{2}\left(\cancelto{0}{H_x^{\ 2}} - \cancelto{0}{H_y^{\ 2}} - H_z^2\right)$$

4. a) Set the field in the air gap as H_0 for $0 < x < \delta$ and $H_0{}'$



For $\delta < x < l$

$$\oint_C \overline{H} \bullet \overline{dl} = \int_V \overline{J} \bullet \overline{da} = NI$$

$$H_0 = \frac{N_0\, I_0}{g} \quad , \quad H_0{}' = \frac{N_0\, I_0 + Ni}{g}$$

b) Set the flux linking the N_o turn coil λ_1

and flux linking the N turn coil, λ_2

$$\lambda_1 = N_0\, \mu_0 \left[\frac{N_0 I_0}{g}\, 2\pi R\, \delta + \frac{N_0\, I_0 + Ni}{g}\, 2\pi R\left(l - \delta\right)\right]$$

$$\lambda_2 = N\mu_0 \left[\frac{N_0\, I_0 + Ni}{g}\, 2\pi R\left(l - \delta\right)\right]$$

c) $$v = \frac{d\lambda_2}{dt} = \frac{d\lambda_2}{d\delta}\frac{d\delta}{dt} = N\mu_0\,\omega\frac{N_0 I_0 + Ni}{g}\;2\pi R\,\delta_0 \sin(\omega t)$$

d) $$\bar{f} = \int \bar{I}\times\bar{B}\;dl = i\left(\frac{\mu_0 H_0 + \mu_0 H'_0}{2}\;2\pi R\right)\bar{i_x}$$

$$f_x = -2\pi R\,\mu_0\,Ni\left(\frac{2N_0 I_0 + Ni}{2g}\right)$$

Co-energy method

$$dW' = \lambda_1\,di_1 + \lambda_2\,di_2 + f_e\,dx$$

$$W' = \int_{\substack{0\\ i_2=0}}^{I_0} \lambda_1 di_1 + \int_{\substack{0\\ i_1=I_0}}^{i} \lambda_2 di_2$$

$$= N_0\,\mu_0\left[\frac{N_0 I_0^2}{2g}\;2\pi R\,\delta + \frac{N_0 I_0^2}{2g}\;2\pi R\;(l-\delta)\right]$$

$$+\; N\mu_0\left[\frac{2N_0 I_0\,i + Ni^2}{2g}\;2\pi R\,(l-\delta)\right]$$

$$f_x = \left.\frac{dW'}{d\delta}\right|_{I_0,\,i} = -2\pi R\,N\mu_0\,i\left(\frac{2N_0 I_0 + Ni}{2g}\right)$$

5. a) $$v = \frac{C_+ + C_-}{2}\;,\quad \frac{T}{\sqrt{\rho E}} = \frac{C_- - C_+}{2}$$

Initial condition:

$$0 < x < a\;,\quad C_+ = -\frac{T_m}{\sqrt{\rho E}}\;,\quad C_- = -\frac{T_m}{\sqrt{\rho E}}\;,\quad T_m = T\;,\quad v = 0$$

$$-a < x < 0\;,\quad C_+ = C_- = 0\;,\quad T = 0\;,\quad v = 0$$

Boundary Conditions: at $x = a$ Fixed end , $v = 0$, $C_- = -C_+$

at $x = -a$ Free end , $T = 0$, $C_+ = C_-$

1. $C_+ = -\frac{T_m}{\sqrt{\rho E}}$, $C_- = \frac{T_m}{\sqrt{\rho E}}$, $v = 0$, $T = T_m$

2. $C_+ = 0$, $C_- = 0$, $v = 0$, $T = 0$

3. $C_+ = 0$, $C_- = 0$, $v = 0$, $T = 0$

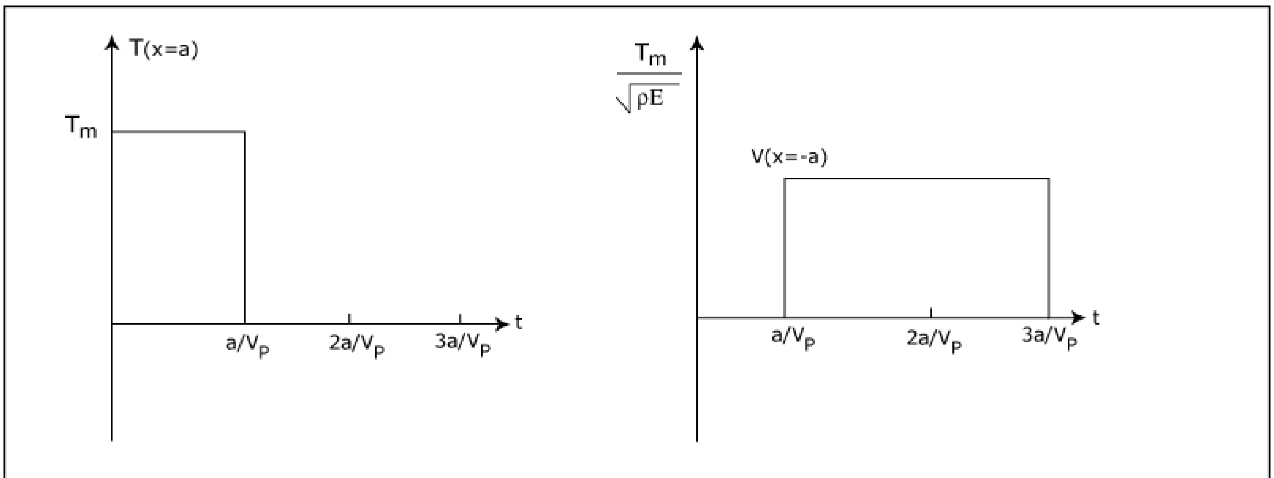
4. $C_+ = 0$, $C_- = 0$, $v = 0$, $T = 0$

5. $C_+ = \frac{T_m}{\sqrt{\rho E}}$, $C_- = \frac{T_m}{\sqrt{\rho E}}$, $v = \frac{T_m}{\sqrt{\rho E}}$, $T = 0$

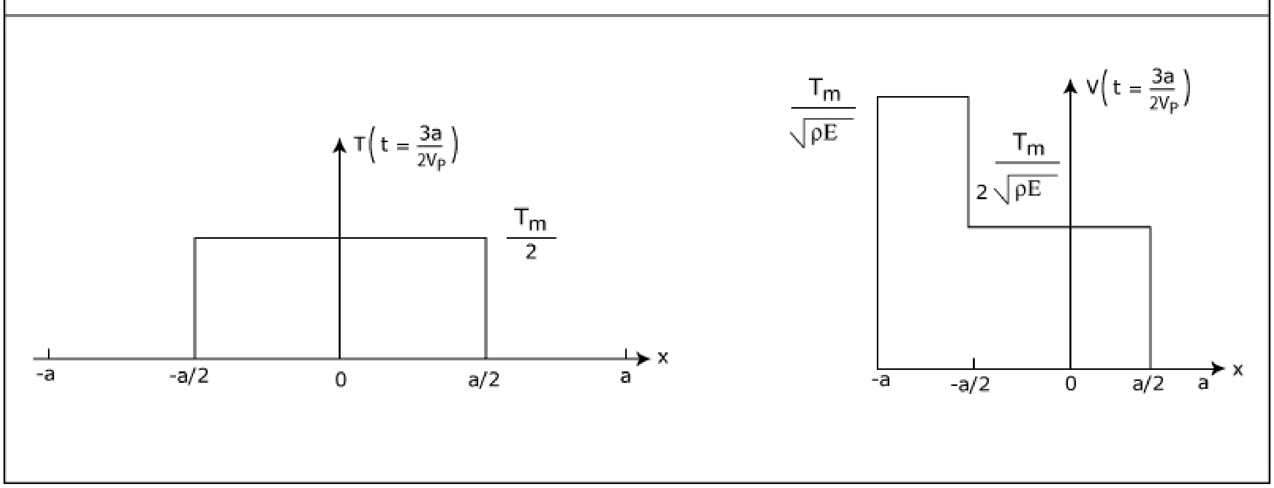
6. $C_+ = \frac{T_m}{\sqrt{\rho E}}$, $C_- = \frac{T_m}{\sqrt{\rho E}}$, $v = \frac{T_m}{\sqrt{\rho E}}$, $T = 0$

7. $C_+ = 0$, $C_- = \frac{T_m}{\sqrt{\rho E}}$, $v = \frac{T_m}{2\sqrt{\rho E}}$, $T = \frac{T_m}{2}$

b)



c)



6. a) $E_1 = \dfrac{V_1}{a-\xi} \approx \dfrac{V_1}{a}\left(1+\dfrac{\xi}{a}\right)$

$$E_2 = \frac{V_0}{a+\xi} \approx \frac{V_0}{a}\left(1-\frac{\xi}{a}\right)$$

b) $\dfrac{F_z}{\text{Area}} = T_{zz_1} - T_{zz_2}$, $T_{zz} = \dfrac{\varepsilon_0}{2}\left(E_z{}^2 - E_x{}^2 - E_y{}^2\right)$

$$= \frac{\varepsilon_0}{2}\left(E_{z_1}{}^2 - E_{z_2}{}^2\right)$$

$$= \frac{\varepsilon_0}{2}\left[\left(\frac{V_1}{a}\right)^2\left(1+\frac{\xi}{a}\right)^2 - \left(\frac{V_0}{a}\right)^2\left(1-\frac{\xi}{a}\right)^2\right]$$

$$= \frac{\varepsilon_0}{2}\left[\left(\frac{V_1}{a}\right)^2\left(1+2\frac{\xi}{a}\right) - \left(\frac{V_0}{a}\right)^2\left(1-2\frac{\xi}{a}\right)\right]$$

c) $\sigma_m \dfrac{\partial^2\xi}{\partial t^2} = S\dfrac{\partial^2\xi}{\partial x^2} + \dfrac{F_z}{\text{Area}} - \sigma_m g$

$$= S\frac{\partial^2\xi}{\partial x^2} + \frac{\varepsilon_0}{2}\left[\left(\frac{V_1}{a}\right)^2\left(1+2\frac{\xi}{a}\right) - \left(\frac{V_0}{a}\right)^2\left(1-2\frac{\xi}{a}\right)\right] - \sigma_m g$$

d) $\left.\dfrac{F_z}{\text{Area}} - \sigma_m g\right|_{\xi=0} = 0$

$$\frac{\varepsilon_0}{2}\left(\left(\frac{V_1}{a}\right)^2 - \left(\frac{V_0}{a}\right)^2\right) - \sigma_m g = 0 \Rightarrow V_1 = \left[V_0{}^2 + \frac{2a^2\sigma_m g}{\varepsilon_0}\right]^{1/2}$$

$$\frac{F_z}{\text{Area}} - \sigma_m g = \frac{\varepsilon_0}{2}\left[\left(\frac{V_1}{a}\right)^2 \cdot 2\frac{\xi}{a} + \left(\frac{V_0}{a}\right)^2 \cdot 2\frac{\xi}{a}\right]$$

$$= \frac{\varepsilon_0}{a}\left[2\left(\frac{V_0}{a}\right)^2 + \frac{2\sigma_m g}{\varepsilon_0}\right]\xi = \frac{\varepsilon_0}{a^3}\left[V_1^2 + V_0^2\right]\xi$$

e) if $\xi(x,t) = \text{Re}\left[\hat{\xi} e^{j(\omega t - kx)}\right]$

$$-\sigma_m \omega^2 \xi = -Sk^2 \hat{\xi} + \frac{\varepsilon_0}{a}\left[2\left(\frac{V_0}{a}\right)^2 + \frac{2\sigma_m g}{\varepsilon_0}\right]\hat{\xi}$$

$$\omega^2 = k^2 \frac{S}{\sigma_m} - \frac{2\varepsilon_0}{a\sigma_m}\left[\left(\frac{V_0}{a}\right)^2 + \frac{\sigma_m g}{\varepsilon_0}\right] = k^2 \frac{S}{\sigma_m} - \frac{\varepsilon_0}{a^3 \sigma_m}\left[V_1^2 + V_0^2\right]$$

$$\omega^2 = v_P^2 k^2 - \omega_c^2 \quad , \quad v_P^2 = \frac{S}{\sigma_m} \quad , \quad \omega_c^2 = \frac{2\varepsilon_0}{a\sigma_m}\left[\left(\frac{V_0}{a}\right)^2 + \frac{\sigma_m g}{\varepsilon_0}\right]$$

$$= \frac{\varepsilon_0}{a^3 \sigma_m}\left[V_1^2 + V_0^2\right]$$

f) at $x = 0$ $, \xi(x = 0, t) = 0$

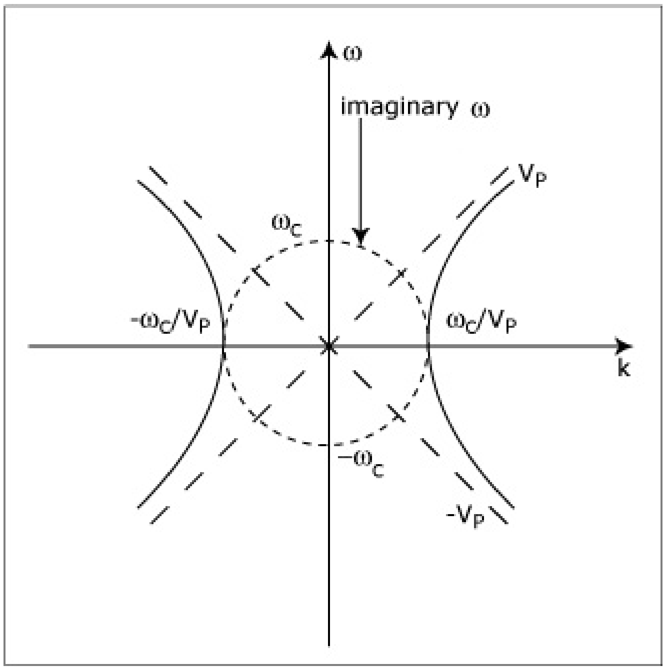
$$x = L \ , \ \xi(x = L, t) = 0 \ ; \ k_0 = \sqrt{\frac{\omega^2 - \omega_c^2}{v_P^2}}$$

$$\xi(x,t) = \text{Re}\left\{\left(\hat{\xi}_1 e^{-jk_0 x} + \hat{\xi}_2 e^{jk_0 x}\right) e^{j\omega t}\right\}$$

$$\hat{\xi}_1 + \hat{\xi}_2 = 0 \Rightarrow \hat{\xi}_1 = -\hat{\xi}_2; \quad \hat{\xi}_1\left(-2j\sin k_0 l\right) = 0 \Rightarrow k_0 = \frac{n\pi}{l}, n = 1, 2...$$

g) First unstable when $k = \frac{\pi}{l} = \frac{\omega_c}{V_P}$

$$\omega_c^2 = V_P^2 \left(\frac{\pi}{l}\right)^2 \Rightarrow V_1^2 + V_0^2 = \left(\frac{\pi}{l}\right)^2 \frac{Sa^3}{\varepsilon_0}$$


ω
imaginary ω
V_P
ω_c
$-\omega_c/V_P$
ω_c/V_P
k
$-\omega_c$
$-V_P$